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The Influence of Risk Taking on Student Creation of Mathematical Meaning: Contextual Risk Theory

Erin Nicole Houghtaling
Brigham Young University - Provo

THE INFLUENCE OF RISK TAKING ON STUDENT CREATION OF MATHEMATICAL MEANING: CONTEXTUAL RISK THEORY

by

Erin Houghtaling Badger

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of a thesis submitted by

Erin Houghtaling Badger

This thesis has been read by each member of the following graduate committee and by majority vote has been found to be satisfactory.

Date	Janet Walter, Chair
Date	Hope Gerson
Date	Robert Speiser

BRIGHAM YOUNG UNIVERSITY

As chair of the candidate's graduate committee, I have read the thesis of Erin Houghtaling Badger in its final form and have found that (1) its format, citations, and bibliographical style are consistent and acceptable and fulfill university and department style requirements; (2) its illustrative materials including figures, tables, and charts are in place; and (3) the final manuscript is satisfactory to the graduate committee and is ready for submission to the university library.

Date	Janet Walter Chair, Graduate Committee

Accepted for the Department

Date	Steve Williams Chair, Department of Mathematics Education

Accepted for the College

Date	Thomas W. Sederberg Associate Dean, College of Physical and Mathematical Sciences

ABSTRACT

THE INFLUENCE OF RISK TAKING ON STUDENT CREATION OF MATHEMATICAL MEANING: CONTEXTUAL RISK THEORY

Erin Houghtaling Badger

Department of Mathematics Education

Master of Art

The primary concerns of mathematics educators are learning and teaching mathematics. It is, therefore, natural to ask “what implications and benefits might there be if learning were perceived as a risk-taking event?” (Atkinson, 1957, p. 266). The underlying motivation of this study is to analyze the risks students take in the mathematics classroom and how risk influences student creation of meaning and development of understanding. I define risk in the mathematics classroom to be any observable act that entails uncertain outcome. The research presented here focuses on a table of four students: Andrew, Carina, Kam, and Mark as they grapple with the mathematical uncertainties inherent in the Ticket Line Task. In analyzing student work and development of mathematical understanding, I identify risks that students take and the benefits they claim result from doing so. Contextualized Risk Theory (CRT) is introduced to improve our understanding

of the risks students take in learning mathematics in a student-centered classroom where students exercise personal agency in mathematical problem solving. Findings include characterization of risks these students took, significant student mathematical activity, student enjoyment of their work, student development of personal understanding of purposes and meanings of specific mathematics, and students achieving mathematical success as defined by the researcher and the participants.

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Chapter 1: Introduction

The primary concern of mathematics educators should be that of learning and teaching mathematics. It is, therefore, natural to be interested in methods that help students effectively learn mathematics. Atkinson (1957) asked, "what implications and benefits might there be if learning were perceived as a risk-taking event?" (p. 266). Atkinson's guiding question provides the underlying motivation of this study: to analyze the risks students take in the mathematics classroom and the influence that risks have on student creation of meaning and development of understanding. The purpose of this study is to build theory improving our understanding of the risks students take in learning mathematics, and suggest implications for traditional and reform-oriented classrooms. I define risk to be any observable act that entails uncertain outcome.

This research centers on college honors calculus students as they engage in collaborative problem-solving of one particular task. These students were part of a teaching experiment in which two instructors team-taught throughout the semester, presenting students with tasks designed to elicit the need for specific mathematics rather than as application or practice problems. The research presented here focuses on a table of four students, Andrew, Carina, Kam, and Mark, as they grapple with the mathematical uncertainties inherent in the Ticket Line Task (Figure 4.4.1). In analyzing student work and development of mathematical understanding, I will identify risks students took and the benefits they claimed would result from doing so. I also highlight student enjoyment in their mathematical work and development of personal understanding of purposes and meanings of specific mathematics.

Chapter 2: Theoretical Perspective

My theoretical perspective lies on the philosophy that learning is a complex, multi-faceted concept. We do not know all of the elements that constitute learning, but emotional and cognitive development are two components particularly relevant to risk taking (Meyer & Turner, 2002; Sullivan, Tobias, & McDonough, 2006). The emotional aspect of student growth is "integral for understanding" (Meyer & Turner, 2002, p. 108) students' learning processes and the decisions they make in the classroom. These decisions, according to Sullivan et al. (2006) can be "to act or to avoid acting, whether being driven by desire or fear" (p. 82). The choice to act contributes to a framework through which I analyze risk.

If risk is emotionally associated, then we must look at what it is students desire or fear from their actions. In general, students associate specific actions with their respective potential results (Atkinson, 1957; Clifford, 1991; Hmelo-Silver, 2004; Koller, Baumert, & Schnabel., 2001; Middleton & Spanias, 1999; Sternberg, 2006; Sullivan et al., 2006). Students evaluate a given potential behavior, look at the opportunities for success and the benefits of taking such an opportunity, and then choose to act upon that information. As individual students perceive opportunities for success, they are more likely to engage in a higher level of risk taking. Because students cannot know the actual benefits of particular actions until after that action is taken, feedback from teachers and peers provides students opportunity to modify their actions, increases self-confidence, and encourages students to take additional risks in the future (Atkinson, 1957; Koller et al., 2001).

In order for students to value the benefits of risk-taking, certain conditions of the mathematics classroom must lend themselves to risk. The classroom environment must be one of safety and companionship (Atkinson, 1957; Clayton, 2007; Francisco, 2005; National Council of Teachers of Mathematics [NCTM], 2007; Ryan & Deci, 2000; Sternberg, 2006). Students should feel safe discussing their work with one another in small and large group settings. They should be able to recognize and value discussion of incorrect answers and solution methods, as well as the approaches that lead to "correct" solutions. Students should be given appropriately challenging tasks that are elaborate, carry real-life meaning or implications, require integration of previous knowledge, and place demands on students' problem-solving and reasoning skills (Atkinson, 1957; Clifford, 1991; Francisco & Maher, 2005; Hiebert et al., 1996; Hmelo-Silver, 2004; Middleton, 1995; Middleton & Spanias, 1999; NCTM, 2000; Noddings, 1989; Sternberg, 2006). The classroom setting and task implemented in this research are aligned with these principles and thus provide a rich environment for studying risk and risk-taking behaviors of university honors calculus students.

Prior research suggests that for the classroom to have particularly rich environmental conditions present, teacher and students each have a niche or role to fill, must be aware of that role, and fulfill it. Students are expected to participate in the problems and tasks discussed during class which the teacher facilitates; students should feel comfortable and safe approaching the teacher with questions or concerns, and teachers should provide opportunity for students to explore and create mathematics (Clayton, 2007; Davis, 1955; Hiebert et al., 1996; Hmelo-Silver, 2004; Kieran, 1998; Meyer & Turner, 2002; Middleton, 1995). Researchers have also claimed that social interaction forms the processes wherein learning occurs and can be assessed (Brown, 1996; Francisco, 2005; Guthrie, 1997; Hiebert et al., 1996; Hmelo-Silver, 2004; Meyer &

Turner, 2002; Sullivan et al., 2006). My theoretical perspective takes a somewhat different stance than the research cited.

I believe the aforementioned roles of teachers and students are too constraining and that the relationship between the actions of teachers and their students is fluid. As the roles of teacher and students become less stratified, students are enabled to become problem posers and can hypothesize, challenge, create, and think outside of the boundaries traditionally imposed on them. Such students would be more prone to engaging in intellectual risk taking than their typical counterparts.

Another social aspect crucial to this research is the need to realize that this study is based on observations of student interactions in an honors calculus classroom. The observed social processes are therefore fundamental to interpreting the risks students take in the mathematics classroom and the outcomes perceived by students upon taking such risks.

Chapter 3: Literature Review

3.1 Definitions of "Risk"

Existing literature includes various definitions for risk or risk-taking behavior. Tholkes (1998) defined risk as "the potential to lose something of value" (p. 24) and claimed that people must choose whether or not they are willing to take risks. Clifford (1991) also connotes risk taking as a negative event, calling it a "decision situation…characterized by a lack of certainty and the prospect of loss or failure" (p. 264).

However, risk does not need to be associated with only negative outcomes. If we choose to do so, we can view risk through a 'glass half-full' lens and see risk not only as action with an admitted probability of failure, but more importantly as action with likelihood for success. The lens through which we define risk is important, because "the way in which risk taking behaviour is defined affects conclusions drawn about such behaviours" (Atkins, Leder, O'Halloran, Pollard, & Taylor, 1991, p. 306). Therefore, in this study risk is examined as it contributes to student learning or other unforeseen, positive implications of student risk-taking in the mathematics classroom.

3.2 Need for Uncertainty

Risk derivates from uncertainty in a given situation. Dewey (1920) claimed that "the origin of thinking is some perplexity, confusion, or doubt" (p. 12). Hmelo-Silver (2004) discussed the necessity for all students to grapple with the uncertainty involved in classroom activities. Hiebert et al. (1996) also emphasized the value in allowing students to deal with situations involving difficulty and doubt. NCTM (2007) asserted that teachers should provide a "climate for students to take intellectual risks in raising questions and formulating conjectures" (p. 40). Uncertainty and confusion require students to take some sort of risk because the outcome is unknown and therefore beyond their comfort level. In this research, particular attention was given to episodes in which students admit difficulty, uncertainty, or doubt.

3.3 Costs and Benefits

In situations involving unknown outcomes, learners often weigh the costs and benefits associated with taking the risk involved (Tholkes, 1998; Guthrie, 1997; Sternberg, 2006; Middleton & Spanias, 1999; Clifford, 1991; Mao, 1991). When learners decide that the value or benefits of participating in a given activity exceed the possible negative outcomes, it is probable that they will proceed to act accordingly. Such a situation has been described as providing "acceptable risk," "moderate risk," "optimal challenge," or "appropriate difficulty" (Guthrie, 1997; Meyer & Turner, 2002; Hiebert et al., 1996). Mao (1991) hypothesized that risks are not generally taken in the mathematics classroom because the benefits associated with doing so do not outweigh the costs.

In this study I demonstrate that under certain conditions, risks *are* taken in the mathematics classroom and students value the associated benefits of doing so enough to sufficiently make risk taking a worthwhile endeavor. Characteristics of the classroom environment that allowed students in this study to engage in risk-taking behaviors and student actions characterized as risk taking will be presented for discussion.

3.4 Success

Because risk is associated with the costs and benefits it provides, it follows that there are various types of risks dependent on the many factors involved (Tholkes, 1998; Guthrie, 1997; Clifford, 1991). Some of these factors include self-confidence, experience, group behavior, societal pressure, the need for challenge or success, and risk-taking as a cultural value (Guthrie, 1997). Koller, Baumery, & Schabel (2001) claimed that as students view themselves as increasing in competence, they are more motivated and willing to engage in risk-taking. Atkinson (1957) and Clifford (1991) also described the positive, linear relationship between an increase in success in the classroom and an increase in the amount of risks students take. Clifford warns of the danger that "we are too culturally addicted to success to sell students on the notion of moderate intellectual risk taking" (p. 274) and that teachers do not foster risk-taking conditions in the classroom because of the belief that taking risks is "inherently aversive" (p. 274) to learning. Hmelo-Silver (2004) argued the importance of risk even when students fail; in grappling with uncertainty students are better able to create meaning for mathematics.

Because risk and success are so intimately related, how society chooses to measure success helps determine its members' willingness to engage in risk-taking practices. In this study, I examine the value of risk in a mathematics classroom where I define success to be students' progress toward solutions of complex problems and tasks, and increase in student understanding of the purposes and meanings of mathematics.

3.5 Motivation

Some studies have claimed that risk may be an expression of student motivation (Hannula, 2006; Sullivan et al., 2006). This is based on the premise that emotions provide a direct link to motivation and that risk is itself an emotional act. However, other studies suggest that motivation occurs on many levels and orientations (Ryan & Deci, 2000; Walter, Hart, & Gerson, 2009), and risk could be considered one such orientation. Walter, et al. (2009) discuss motivation as a complex, multi-faceted element comprising an individual’s desire, power, and tendency to act. Accordingly, students may find themselves motivated on several levels and exhibit intellectual passion and multiple tendencies in action to choose among simultaneously existing motivations. Although motivation may be a possible lens through which to view risk taking behaviors and events, I am intrigued by the implications for learning that arise when learning is perceived as a risk-taking event (Atkinson, 1957, p. 266).

3.6 Social Aspects

My perspective builds on ideas of social constructivism: that mathematical meaning is socially constructed, and that personal aspects of learning are inextricably connected to the social aspects (Francisco, 2005; Sullivan et al., 2006; Brown, 1996; Guthrie, 1997; Davis, 1955; Meyer & Turner, 2002). The social and personal aspects interact as a system where students employ both modalities to develop mathematical understanding.

The students observed in this study collaborated in small groups attempting to make sense of and come to solutions of the given task. Sullivan et al., (2002) argued that members of a group either inhibit or enhance one another’s opportunity for learning and growth. Guthrie (1997) claimed that social risk is a “significant source of fear” (p. 215) that is often stronger than other types of risk. Guthrie discussed the swaying effect of peer pressure and a scenario called “risky shift” where groups make more risky decisions because each member places some of his personal responsibility on the “others,” thereby relieving each member of some of the personal costs and benefits involved. The particular group dynamics among the participants and participant responses to a mathematics perspectives survey are discussed in the methods chapter.

Francisco (2005) claimed that self-expression benefits the learning process; this self-expression must take form in some social setting, even if the audience is oneself. Brown (1996) claimed that meaning is produced in discourse. As such it is important to observe the social and discursive actions in the classroom to better understand the meanings of mathematics being learned. Hmelo-Silver (2004) called for more research on learning that occurs in social settings and that is problem based. My research responds to that call.

3.7 Model for Mathematical Growth and Understanding

Pirie and Kieren (1994) developed a model for mathematical growth and understanding and claimed that learning is not linear: students experience various layers of understanding for various durations at different moments in their work. As students work on a problem, they experience moments of misunderstanding or of grappling with the unknown; at such moments they return to previous levels of understanding, or "fold back" to former levels and build upon what is already understood, ultimately yielding an increased growth of mathematical understanding. Hmelo-Silver (2004) also claimed that as students with only simple understanding discuss a problem with others, they activate prior knowledge and can then move forward in the development of their understanding. Thus social interaction can spur this "folding back" as students build mathematical meaning.

Pirie and Kieren (1994) also discussed "expression" and "action" as the cooperative means by which students move among levels of understanding and thus experience mathematical growth and understanding.

Pirie and Kieren's "expression" is similar to what Francisco (2005) discussed as self-expression. Pirie and Kieren's (1994) "action" is briefly described as that which encompasses previous understanding and provides continuity within each inner layer of understanding. As "action" is not given further elaboration, I believe that the "action" enabling students to come to deeper understanding of significant mathematics may be the actual process of taking a risk.

As students take risks and grow in their understanding, they respond to feedback provided by mentors and peers. This feedback is seen as a necessary element of positive risk taking experiences because as feedback increases in value to students, so does risk taking (Atkinson, 1957).

3.8 Need for More Work

In order to continue to understand student learning of mathematics, comprehensive work studying the interaction of emotion, motivation, and cognition is needed (Meyer & Turner, 2002). Several ways in which risk is related to emotion and motivation have been discussed. Here, the focus turns to the links between risk and student cognitive development. Francisco and Maher (2005) and Schoenfeld (1992, 1985) [in Francisco & Maher, 2005, p. 362; Francisco, 2005, p. 67] claimed a need for more research discussing how mathematical thinking, problem solving, and practices in mathematical communities fit together. Clifford (1991) stated that there is a "limited amount of research on academic risk taking" (p. 292) and suggested the possibility and necessity of conducting field research while simultaneously pursuing theory development.

The research presented in this thesis identifies potentially significant academic risk taking behaviors exhibited by students in their work on the Ticket Line Task and contributes potentially important insights of student engagement in risk taking behaviors in the mathematics classroom.

3.9 Research Question

In this study I address the questions: what risks do students take in learning mathematics, and how does taking risks influence student creation of mathematical meaning? I develop Contextual Risk Theory (CRT) to improve understandings of the risks students take in learning mathematics in a student-centered classroom where students exercise personal agency in collaborative mathematical problem solving.

Chapter 4: Methods

4.1 Background of Larger Study

This qualitative study is situated within a broader, longitudinal mixed-methods study conducted as a 3-semester teaching experiment in, and extending beyond, experimental undergraduate honors calculus courses. Two faculty members at a large, private university in the Western United States are conducting a long-term study of students' mathematical work, conversations, invented representations and collaborative efforts based on videotape data collected during classroom sessions. These videotape data are part of the corpus of data within the larger study which also include observation fieldnotes, student work, class assessments, personal interviews, pre- and post-semester surveys of students' perspectives on various aspects of mathematics (student perspectives survey, Appendix A), and mean scores for all university calculus students on a common final exam. The honors calculus students who participated in the teaching experiment were later sent electronic surveys with respect to their subsequent experiences in related courses throughout the remainder of their enrollment at the university.

The larger study was designed to contribute to the field of mathematics education an improved understanding of how students make sense of calculus. Students worked in collaborative settings, with limited lecturing by instructors, on carefully selected, open-response tasks designed to elicit the building of critical mathematics in response to issues that emerged from student explorations and to develop grounded understanding of the fundamental content of calculus.

4.2 Risk Study

This study is a mixed-method analysis of classroom video data of student collaboration on the Ticket Line Task (Figure 4.4.1, TLT) halfway through the first semester of the larger project and of the students' perspectives survey responses collected at the beginning of the semester. Analysis of these data led to the emergence of contextual grounded theory, presented here, regarding student engagement in risk taking during mathematical problem solving.

4.3 Participants

The participants were undergraduates at the university voluntarily enrolled in the honors calculus course. Students worked in groups of five or six, and the focus group of students in this study includes Andrew, Carina, Kameron (Kam), and Mark. At the beginning of the course these students, with the rest of the class, completed a survey on their mathematical perspectives. Relevant details include participant's previous experiences in mathematics courses, qualities of excellent mathematics learners, participants' views of themselves as mathematics learners, what it means to be a successful mathematics learner, what the participants view as being an optimum classroom environment for learning mathematics, and the responsibilities of teachers and students in the honors calculus course. A basic summary of student responses is provided in Table 4.3.1.

Andrew was a junior majoring in bioinformatics. In his responses to the student perspectives survey administered at the beginning of the semester, Andrew identified persistence as his best quality as a math learner: he would work on a problem until he fully understood it. He reported that his weakest quality was the tendency to not have an open mind and to get stuck in his "wrong beliefs" about mathematics. Andrew viewed success as having the “desire to learn and the work ethic to back [that desire] up.” Andrew thought the optimum classroom environment included a basic introduction of the topic, explanations of equations in real-world contexts, and practice with exercise problems. Andrew said he did not like when teachers “attempt to ‘challenge’ their [students’] minds” by giving assignments without providing students with proper preparation and thought that doing so was a waste of time; he wanted teachers to give their students enough information to allow them to complete a challenging task. Andrew’s ideas of a teacher’s responsibilities were aligned with traditionally oriented mathematics classrooms: the teacher discusses main ideas, solves problems, assigns problems, and gives students class-time to work on them. The students’ responsibilities were to “do the homework, read the text, and work until [they] understand what is presented.” It is important to recognize that Andrew claimed to be persistent and open-minded, yet he claimed to not to be open minded; his pre-semester view of the optimum classroom environment is traditionally oriented and different from the experimental honors calculus classes in that the instructors’ focus was to challenge their students and provide students opportunity to develop mathematics through grappling with difficult tasks without prior instruction on solution methods.

Carina was a sophomore majoring in mathematics education. She claimed her strongest qualities were patience and flexibility. She would persist on trying several different ways to solve problems without getting get frustrated with them. Carina said that a successful mathematics learner was able to apply concepts and general patterns of mathematics in the different situations where problems occur. Carina’s optimum classroom environment included peers “willing to learn and share ideas on how to solve the exercises," and she reported that as a student in the course she had a responsibility to be willing to help others and to listen to other members of her group. Carina's pre-semester perspectives are theoretically similar to the ideas that shaped this research and align with the instructors of the course.

Kam was a junior majoring in mechanical engineering. He received A's in the four university mathematics courses he had taken before the course and stated that he enjoyed mathematics more than most of his peers, a quality that "changes math from work to play." Kam reported he tended to give up when he failed, and that being a successful mathematics learner meant understanding principles of math and their applications in a "memorable and useful way." Kam's claimed that an optimum classroom environment was one where teachers and students respect each other and where "multiple approaches are discussed so that each student can decide what works best for them." Kam thought that students should have opportunities to teach mathematics principles in the classroom and stressed the importance of student participation. Kam asserted that students are responsible for being open-minded, attentive, hard-working, and involved in what happens in the classroom. He thought teachers should provide structure for the class, allow interaction and participation, and clearly explain principles in multiple ways. Kam's pre-semester perspectives espouse those of a student-centered classroom where students engage in difficult mathematics and demonstrate that upon entering the honors calculus classroom, Kam's expectations for how the classroom should be run were aligned with the instructors' design.

Mark was a sophomore majoring in bioinformatics/ computer science. He did not complete a survey; therefore his existing perspectives at the beginning of the semester were unknown.

Student Name	Previous Experiences	Participant Math Qualities	Participant Definition of "Success"	Optimum Classroom Environment
Andrew	Junior	Persistence; tendency to not be open-minded	Desire to learn and work ethic to back it up	Basic intro, exploration, then practice
Carina	Sophomore	Patience; flexibility; persistence	Apply concepts in different problem situations	Peers willing to participate so as to learn and share ideas
Kam	Junior; A's in four previous college courses	Enjoys math, gives up when failing	Understanding principles and their application	Mutual respect; student participation
Mark	Sophomore, N/A	N/A	N/A	N/A

Table 4.3.1. Summary of Pre-Semester Student Perspectives Survey.

These students' self-reported perspectives shaped my inferences about risk for these particular students. Andrew, for example, stated that he did not like when teachers would challenge their students and that giving assignments without background information (i.e., the types of activities the instructors in the larger study employed on a regular basis in the Calculus teaching experiment) was a "waste of time." Andrew claimed that as a student his role was to be persistent and work until he understood the material his teacher presented. It was important to recognize the inherent risk for Andrew in participating in a challenging, open-ended task like the TLT as it pushed him outside his self-proclaimed comfort zone. Evidence will be provided in the data demonstrating a shift in Andrew's perspectives on the roles of teacher and student and what a mathematics classroom should look like.

4.4 Mathematical Task

Students were given the Ticket Line Task (Figure 4.4.1), which was adapted by Walter & Gerson[1] in its implementation. As found in Garner (2006), the task was an exercise at the end of a section, whereas Walter and Gerson introduced the task at the beginning of the section to elicit sense making for area under a curve and the need for integration as a problem-solving tool while providing students the opportunity to engage in collaborative problem solving.

The ticket office opens at 8 am to sell student basketball tickets. At that time, there are already 600 students in line and students are arriving at the rate of 500 students per hour. The rate at which students arrive increases steadily until 1 pm, when the rate is 1500 students per hour. The rate at which students arrive then decreases steadily to 0 at 4 pm, when the ticket office closes. The ticket office can serve students at the rate of 1000 students per hour.
a. Sketch a graph of the rate at which students arrive at the ticket office as a function of time.
b. At first, because students are served at a rate greater than the rate at which they arrive, the line decreases in length. Does it disappear before students begin arriving at a rate greater than the rate at which students are being served? About when does the line again form or begin to lengthen?
c. About when is the line the longest? About how many students are in line then? About how long would the last student in line at that instant have to wait to be served?
d. About when does the line finally disappear?
e. About how many students are served by the ticket office that day?
f. Sketch a graph of the length of the line as a function of time.

Figure 4.4.1. The Ticket Line Task.

4.5 Data collection

Here, specific focus is given to analysis of the video data, transcriptions, field notes, and student mathematical work during the 8 ½ hours in class when students worked on or discussed the mathematics that were elicited by the TLT. Data collection

[1] Adapted from Garner, L. E. (2006). *Calculus*. Boston: Pearson Education, 297-298.

for this work comprised gathering existing information on the participants involved and transcribing the video so these data could then be analyzed, annotated, and coded.

4.6 Development of codes

Work on these data began with verbatim transcription of video data, followed by preliminary annotations of video data so as to identify and decipher compelling episodes where students were engaged in legitimate sense-making of difficult mathematics, and to distinguish outcomes of and characteristics surrounding the risks that students took in their work on the TLT. Focus was then given to thirty-eight total minutes of video taken from the first 4 ½ hours of student work. These thirty-eight minutes were selected because they were particularly compelling in terms of providing rich opportunity to study mathematical risk because students were deeply and personally involved in significant mathematical work, displayed great desire to develop personal understanding of the mathematics, and engaged in a variety of distinct behaviors in doing so. Video clips are identified by the day and hour (A signifies the first hour, B signifies the second) corresponding to each class session: the notation Day 2A 45:07.1 denotes that the piece of videotape data comes from the second day (Day 2) of work on the TLT at 45 minutes and 7 seconds (45:07) into the first hour (A). Line by line open coding was then performed for key words and events that emerged in student mathematical work. In particular, attention was given to student computations and mathematical productivity so as to track their progress towards solutions of the task and to allow the telling of the students' mathematical story. This provided opportunity to study how mathematical risks taken affected student learning and progress towards a solution.

Attention was also given to significant changes in the direction of student work so as to ascertain what elements of risk-taking, if any, contributed to such change. Focus was given to the surrounding circumstances of such events so as to identify key components of risk taking.

Particular consideration was given to the instances when students declared or implied an increase or growth in mathematical understanding. This provided insight to the interplay of taking risks and increasing understanding. Because success is defined here as an increase or growth in mathematical understanding, paying attention to the relationship between success and risk might contribute to the claims made by Atkinson (1957) and Clifford (1991) that a statistically positive, linear relationship exists between an increase in success and an increase in the amount of risks students take. Although Clifford and Atkinson characterize risk pessimistically, they note a relationship between success and risk that seemingly contradicts their pessimistic stance toward risk. The relationship between success and risk found in this study is discussed in further detail in the findings chapter.

Upon completion of open coding, definitions for each code were first refined then compared to one another. Open codes were placed into groups based on similar properties. Examples of some groupings follow: open codes that seemed to correspond with student emotion were placed together in one group; codes that dealt with results of student computations were grouped; codes for the general mathematical ideas discussed by the students were placed into another group; codes that characterized the actions students took in grappling with the mathematical uncertainty of their work were placed into another group.

Upon completion of the initial grouping of open codes, each group was compared to one another so as to combine or split groups and ultimately refine the definitions of each group of codes. The resulting seven categorical groups were: student references to the *task*, the *mathematical concepts* discussed among students in their progress towards a solution of the TLT, the specific equations, graphs, and other mathematics students developed in their solution of the TLT (*mathematics from the task*), student behaviors characterized as elements contributing specifically to *mathematical risk taking*, student efforts of *collaboration* in their work on the task, student expressions of *emotion* exhibited during their work on the TLT, and the *speaker* at each unit of analysis. Further elaboration of the categorical groups, including abbreviations used and the open codes that comprise each group are provided in the data and analysis chapter.

Each open code was tagged with the category to which it belonged, and the resulting axial codes were then compared to videotape data to ensure accurate representation of student activities and behaviors. The emergence of the codes and abbreviated notations will be detailed in the data and analysis chapter.

Episode and keyword reports, detailing the frequency and duration of the use of each code were run (Appendix C) so as to contribute quantitatively to the results of this study.

Chapter 5: Data and Analysis

In the data and analysis chapter, the first section provides an overview narrative of particularly compelling pieces of student work on the Ticket Line Task where student behaviors seemed, upon initial analysis, to be illustrative of risk taking scenarios. Specifically, the narrative outlines how students proceeded to build a solution to parts A, B, and F of the Ticket Line Task. The second section details the emergence of grounded codes through presentation and analysis of compelling, selected episodes.

5.1 Initial Student Work

Initial work on the TLT included student interpretation and organization of the information provided in the task though individual work. Mark created a chart (Figure 5.1.1) demonstrating the change in rate of people arriving over the course of the day (Day 1B, 35:14). The task informed students that at 8:00 am 600 people were already in line, and more were arriving at a rate of 500 people per hour. This rate of people arriving increased steadily from 500 people per hour until 1:00 pm when people were arriving at a rate of 1500 per hour. From 1:00 until 4:00 pm, the rate of people arriving decreased steadily from 1500 per hour to 0 people per hour at closing time, 4:00 pm. The task also informed students that the ticket office could serve people at a rate of 1000 people per hour throughout the day. As Mark progressed over the course of the 4 ½ hours spent on the task, he occasionally referred to his chart to remind himself of the information provided in the TLT.

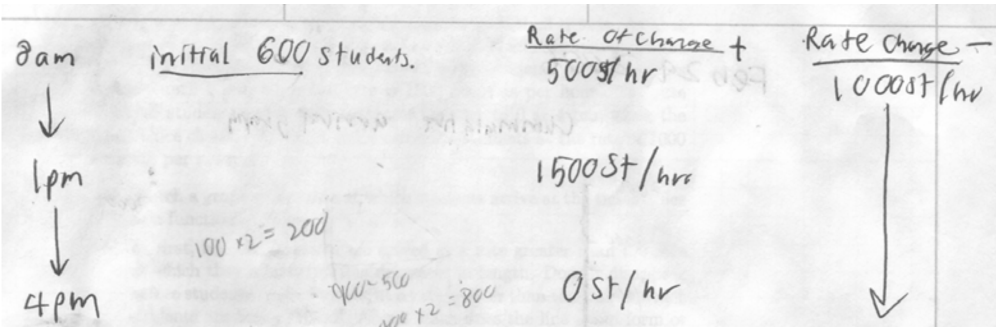


Figure 5.1.1. Mark’s Organizational Chart.

5.2 Part A, Day 1

Each student independently worked on part A of the task (Day 1B, 33:26-44:40) to create a graph of the rate at which people arrive at the ticket office as a function of time. To create this graph, the students started with the information provided in the task, that at 8:00 am (x=0) the rate of people arriving per hour is 500, at 1:00 pm (x=5) the rate of people arriving is 1500, and at 4:00 pm (x=8) the rate of people arriving is 0. The students reasoned that because the rate arriving increases (from 8:00 am until 1:00 pm) and decreases (from 1:00 until 4:00 pm) steadily, they could plot the points (0, 500), (5, 1500), and (8, 0) and connect the points to create a graph of the rate of people arriving. Kam’s graph is representative of what the others also developed, but was particularly neat and legible, thus it is provided in Figure 5.2.1. In this phase of the task students are grounding their work in understanding that existed previous to work on the task beginning to process new information they gather from the task.

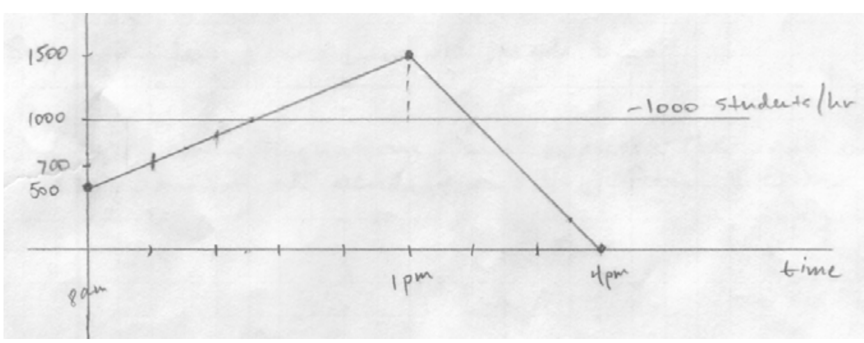


Figure 5.2.1. Graph of the Rates of People Arriving and Leaving.

5.3 Part B, Day 1

Kam then called the group together to discuss part B, which asked students whether they thought the line would disappear before the rate of people arriving at the ticket office was greater than the rate of people being served. He proposed that using the information provided in the task, they could actually create equations representing the rate of people arriving from 8:00 am to 1:00 pm and from 1:00 until 4:00 pm (Day 1B, 46:42 – 48:34). Kam and Mark developed equations to represent the rate of people arriving from 8:00 am until 1:00 pm, and from 1:00 pm until 4:00 pm (Day 1B, 48:34 – Day 2A, 16:56). Their work is provided momentarily because while they were working, Andrew introduced a new and important idea that would lead him to take mathematical risks (Day 1B 48:51).

Without knowing whether others would understand how finding the area under the curves representing the rates of people arriving might contribute to the final solution, Andrew proposed that they could use geometry to find the area under the curve of people arriving for 8:00 am until 1:00 pm and for 1:00 pm until 4:00 pm, and the area under the curve from $x=0$ until $x=n$ would represent the number of people in line at any given time $x=n$ (Day 1B, 48:51). Andrew and Carina declared their intent to find the area of the region above $y=500$ (Figure 5.3.1) and add it to the area of the trapezoid below $y=500$ (Figure 5.3.2), and claimed the summation of the areas would be the number of people who got in line at the ticket office throughout the course of the day.

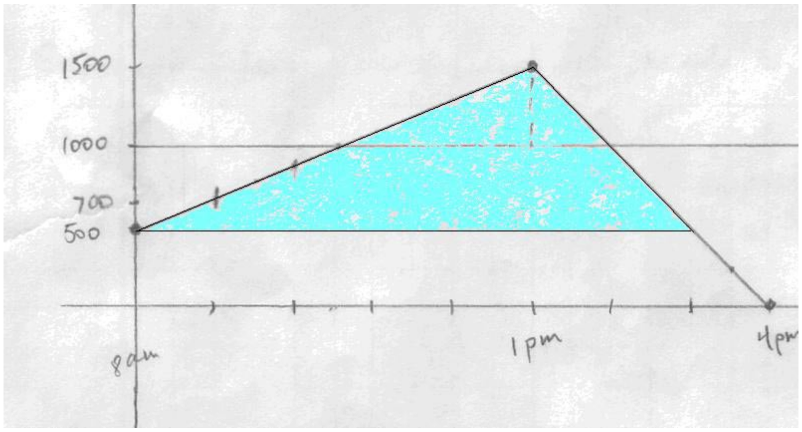


Figure 5.3.1. Triangle above *y=500.*

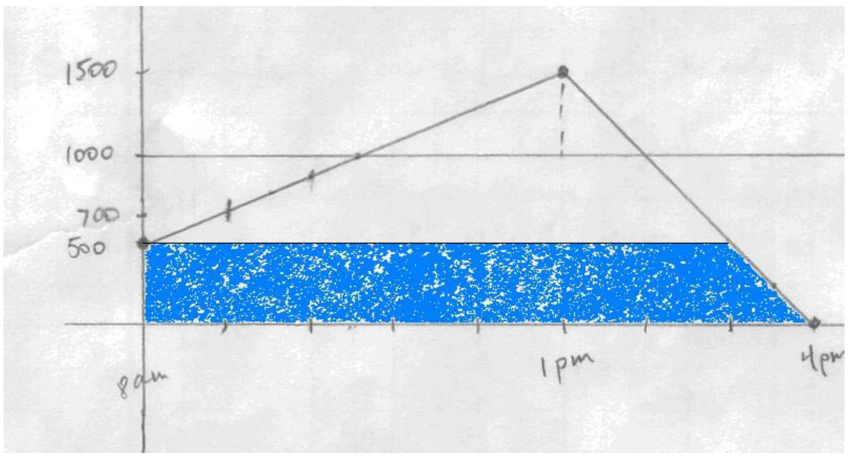


Figure 5.3.2. Trapezoid below *y=500.*

While Carina and Andrew worked to find the shaded areas under the curves using geometric formulas for area of a triangle (Day 1B, 48:58 – 51:49), Mark and Kam worked to find equations representing the linear functions of the rate of students arriving.

Mark and Kam referred to information provided in the task, namely that at 8 am people were arriving at a rate of 500 people per hour and increasing at a steady rate until 1:00 pm, when they were arriving at a rate of 1500 people per hour. They reasoned that since the rate of people arriving increased steadily from 500 to 1500 over a five hour period, the average increase in people arriving was 200 people per hour. Because there were initially 500 people arriving per hour, they were able to create $y_1 = 200x + 500$ to represent the rate of people arriving from 8:00 am to 1:00 pm (Day 1B, 53:03). [Note: the students did not use the notation "y_1=…" but in order to differentiate between the various equations students used, the notation y_1, y_2, etc… is introduced]. Mark and Kam then used the information provided in the task that when the ticket office opened at 8:00 am, there were 600 people in line to determine how many people were in line at 1:00 pm. They later developed $y_2 = -500x + 4000$ to represent the rate of people arriving between 1:00 and 4:00 pm when the ticket office closed in the same manner and using the same reasoning as the development of y_1 (Day 2A, 16:56). It is worth noting that students were not required to find equations for any part of the task, but students chose to recognize mathematical patterns in their work and built equations for representative functions to assist in their own understanding.

Andrew and Carina rejoined Mark and Kam in the development of y_1 and y_2, but Andrew seemed to have difficulty understanding how they derived their equations (Day 1B, 51:46). Instead, he kept returning to the idea of finding the area under the curves y_1 and y_2 and expressed great desire to find one single function representing the total number of people in line at any time from 8:00 am until 4:00 pm, rather than various equations for the rates of people arriving or leaving during various time periods throughout the day. Andrew's persistence and confidence in his idea, an idea that was not upheld by his peers, was grounded in his adherence to the notion that he could find one function using area under the curve that would represent the total number of people in line at the ticket office throughout the day.

5.4 Part A Revisited, Day 2

The second class session of work on the TLT began with students becoming reacquainted with the task. They compared their notes, discussed their work on Part A of the task, and revisited the equations y_1 and y_2, ensuring that every student in the group understood what the equations meant and where each component came from (Day 2A, 0:00 – 34:11).

5.5 Part B Revisited, Day 2

Students resumed work on part B of the task, trying to decide whether the line of people at the ticket office would disappear before 10:30 am (where they determined the rate of people arriving at the ticket office to be greater than the rate of people being served by the ticket office).

Kam's approach to Part B was to use average rate of people's arrival at the ticket office (Day 2A, 34:54 – 37:44). Kam explained to Mark that he knew that from 8:00 until 9:00 am, the rate of people arriving would increase from 500 to 700 people per hour, and decided that the average rate of people arriving during that hour was 600. Kam claimed that in its first hour of operation, 600 people got into line at the ticket office. Since another 600 people were already in line when the ticket office opened, a total of 1200 people got in line for tickets from 8:00 until 9:00 am. Over the course of that hour, 1000 of them were served (given information), and so there were 200 people left in the line. Kam concluded that having 200 people in line meant that the line had not yet disappeared.

Kam then looked at the average rate of people arriving from 8:00 until 10:00 am (Day 2A, 37:44 – 39:26). He reasoned that because people were arriving at a rate of 500 people per hour at 8:00 am, which rate increased steadily to 900 people arriving per hour at 10:00 am, that the average rate of people arriving over the course of the two hours was 700 people per hour. This meant that from 8:00 until 10:00 am, 1400 people arrived. Considering the 600 people already in line when the ticket office opened, Kam claimed that by 10:00 am 2000 people had gotten into line. Since the task said that the office could serve 1000 people per hour, Kam asserted that at exactly 10:00 am, the 2000 people in line will have been served and the line will disappear, thus answering Part B of the task. After listening to Kam's reasoning, Mark expressed agreement in the validity of Kam's work and integrated the average rate of people arriving into his own work on the TLT.

5.6 Andrew's Idea: Area between the Curves

Forty-five minutes into the class period (Day 2A, 45:50), Andrew proposed that they find the area between $y_3 = 1000$ (the rate of people being served per hour) and y_1 (Figure 5.6.1). He reasoned that because the rate of people arriving from 8:00 to 10:30 am was less than the rate of people being served, every person who arrived at the ticket office during that time was automatically served. He recognized the need to account for the 600 people who were already waiting outside of the ticket office doors when it opened at 8:00 am, and argued that the area of the region between y_1 and y_3 (henceforth called Region A) was significant because it represented the number of those 600 people originally in line who have been served.

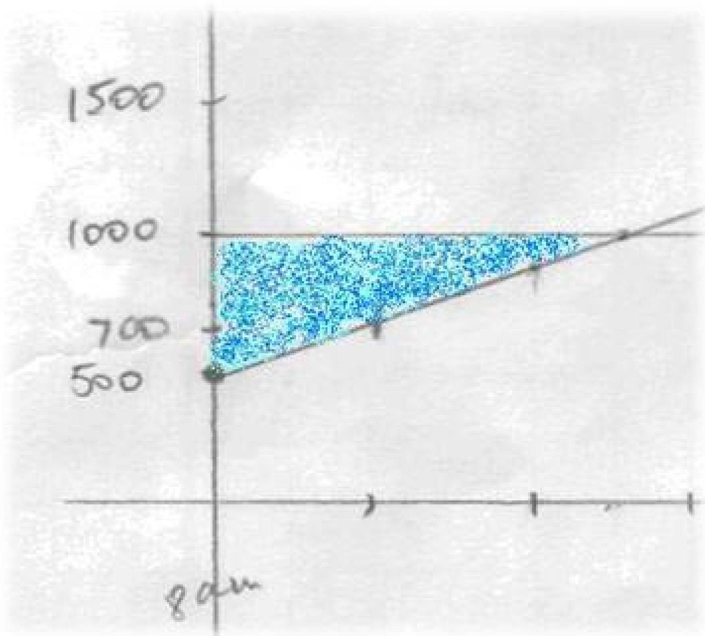


Figure 5.6.1. Region A.

Mark agreed with Andrew (Day 2A, 47:01) and extended the idea to hypothesize that if they can find where to truncate Region A so that its area is 600 (see Figure 5.6.2), they will know when the line disappears.

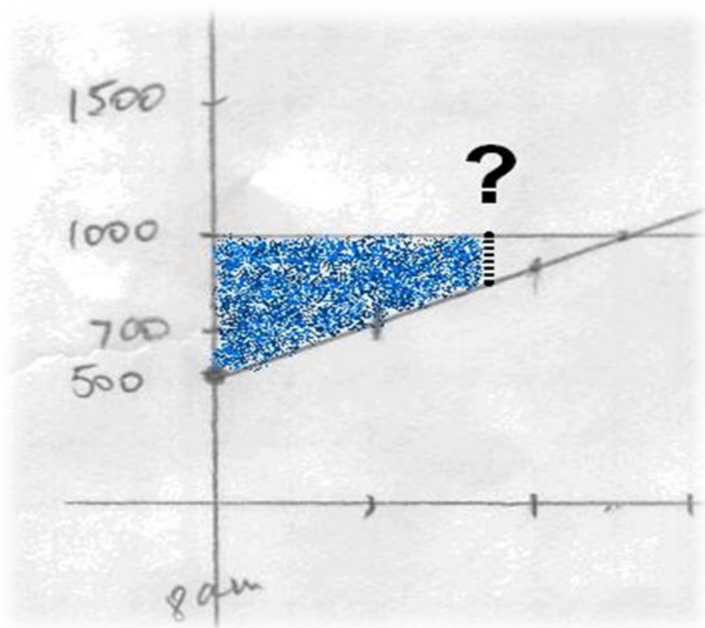


Figure 5.6.2. Truncated Region A.

Andrew and Mark brought to fruition the idea of finding the area of the truncated region by cutting the region into smaller geometric pieces and finding the area of each piece using formulas that they knew to plug in information and find the area (Day 2A, 47:52). In doing so, they soon realized that the area of Truncated Region A would reach 600 at 10:00 am, the same time of day when Kam predicted the line would disappear.

Building off of the idea of using area under the curve where the number of students is known (i.e. finding where 600 students would have been served, or finding where 0 students were in line as in part d of the task), Andrew turned to Carina (Day 2A, 49:09) and asked, "how do you do an integral?" The other members of the group listened to Andrew's question but chose to focus instead on completing the task using areas and average rates of change as they did to answer Part B of the TLT. Meanwhile, Andrew and Carina focused on anti-differentiation, engaging in collaborative problem solving and reasoning together with the purpose of finding one function to represent the total number of people in line at the ticket office at any time throughout the day. Andrew and Carina recognized that such a line needed to account for the people arriving *and* the people being served at the ticket office. After several attempts, they developed $y_7 = 100x^2 - 500x + 600$ by finding what they thought to be the anti-derivative for y_1, or $y_6 = 100x^2 + 500x + 600$, and subtracting the anti-derivative for y_2, or $y_5 = -1000$ (Day 2A, 49:09 – 56:48). Excited with their work, Andrew and Carina used the trace function on a graphing calculator to check and hopefully confirm that what they have done is correct (Day 2B, 0:59 – 12:31). Earlier work (Day 1B, 48:58 – 51:49) had convinced them that from 10:00 until 10:30 am there would be 0 people in line and at 1:00 pm there would be 625 people in line. Much to their dismay, Andrew and Carina discovered that the graphing calculator trace indicated that -25 people would be in line at 10:30 am, and 600 at 1:00 pm.

After brainstorming and searching for possible explanations for the errors, they hypothesized that because the graph of y_6 dips below the x-axis (Figure 5.6.3), the number of people in line from 10:30 am until 1:00 pm had been shifted down by 25 units (Day 2B, 13:19 – 17:08).

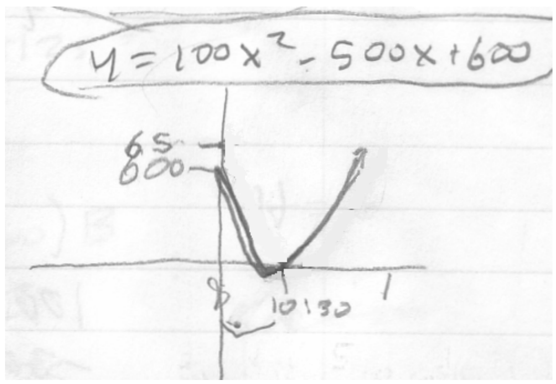


Figure 5.6.3. Andrew's Graph of y_7.

Andrew proposed that instead of having just one equation representing the number of people in line at the ticket office throughout the day, they should create a piece-wise function with different equations to represent the number of people in line at various time periods throughout the day (Day 2B, 15:52). He proposed, for example, that y_7 represented the number of people in line at the ticket office at any time between 8:00 and 10:00 am, and that $y_4 = 0$ represented the number of people in line at the ticket office at any time between 10:00 and 10:30 am. The introduction of such a novel concept that had not been discussed by anyone else in the group constitutes mathematical risk-taking and is elaborated upon further in the "intellectual adventuring" section of the findings chapter. Further exploration of this proposal continued until after 5 total hours on the task when Andrew and Carina developed one piece-wise function and corresponding graph to describe the number of people in line at the ticket office any time of the day (Figure 5.6.4). Upon completing his work, Andrew excitedly exclaimed, "Ba-bam! We figured it out!"

Andrew's function was

$$y = \begin{cases} 100x^2 - 500x + 600 & 0 \le x < 2 \quad or \quad 8:00 - 10:00am \\ 0 & 2 \le x < 2.5 \quad or \quad 10:00 - 10:30am \\ 100x^2 - 500x + 625 & 2.5 \le x < 5 \quad or \quad 10:30am - 1:00pm \\ -250x^2 + 3000x - 8125 & 5 \le x < 7.87 \quad or \quad 1:00 - 3:52pm \\ 0 & 7.87 \le x < 8 \quad or \quad 3:52 - 4:00pm \end{cases}$$

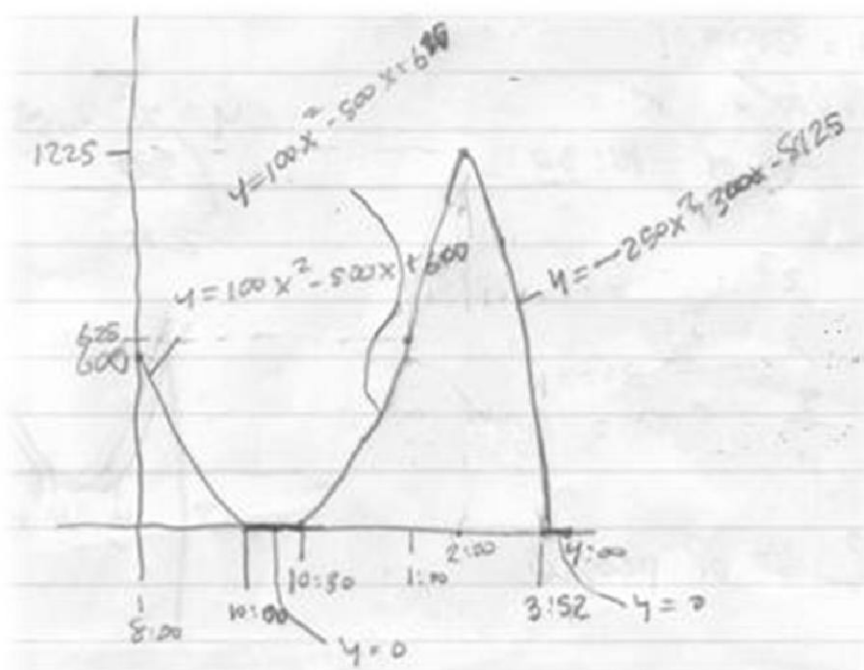


Figure 5.6.4. Andrew's Piecewise Function and Graph for Part F.

Andrew and Carina were the only students in the class to develop a function describing the number of people in line throughout the day at the ticket office. Other students, including Kam, used the average rate of people arriving to create tabular and graphical representations of the number of people in line throughout the day (Figure 5.6.5).

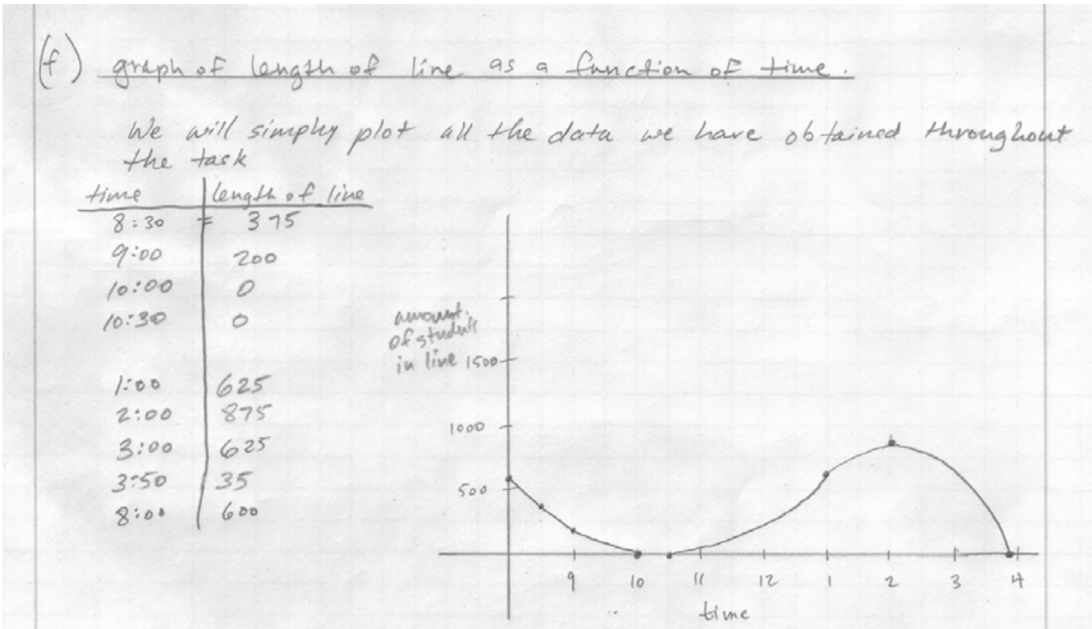


Figure 5.6.5. Kam’s Graph of Part F.

5.7 Emergence of Codes

From a total of 8 ½ hours of video data, representative video excerpts totaling 38 minutes of significant mathematical work by students were selected. The video and annotated transcripts were initially analyzed in an open coding of key words that emerged from video transcription, annotations, participant gestures, etc. I looked for evidence to tell the students’ mathematical story and to describe the methods they used in working on the task, including those methods I suspected were involved in student risk taking. Upon completion of open coding, I created definitions for each code word based on the data from which each code emerged (Appendix B) and then grouped the codes categorically. Seven categories emerged and are detailed below: the student speaker (S), references to the TLT (T), mathematical concept elicited by the task (MC), specific mathematics students created or used in solving the task (MT), observable student emotional responses

(E), methods of student collaboration (C), and behaviors categorized as explicitly relating to mathematical risk-taking (MR).

The speaker codes (S) referred to who was speaking (S for *speaker*). By coding for the student speaking, the frequency and duration of each student's contribution to the conversation at the table could be tracked. The principal players were Mark, Kam, Carina, and Andrew. I coded for the speaker so that I could identify which students seemed to be heavily engaged in taking risks.

The TLT codes (T) referred to some specific aspect of the task, whether in reference to information provided in the task or one of the sub-questions, part A through F. Below is a transcript piece providing an example of some (T) codes (transcript references to the task are bolded):

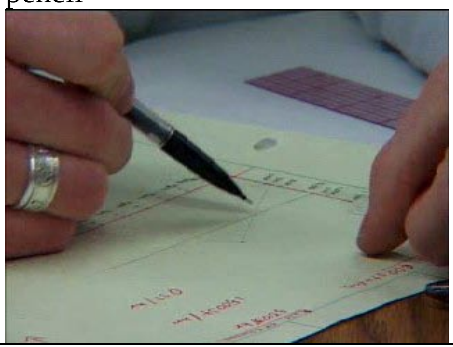
Day 1B 0:46:42.0	Kam	**What'd you guys say for B? Has anybody done B?**	[To the entire group] Part B asks when the line first disappears and when it begins to form/lengthen again	C- compare; **T- part B; T- referral (direct);**
0:46:46.5	Mark	I'm looking at **it** right now.		**T- referral (indirect);**
0:46:49.3	Kam	I think there's gonna be- Do you think the **line's gonna disappear before students begin arriving at a rate greater than the rate at which students are being served**?	Rephrasing **part B** of the task, asking the group whether they think the line of people at the ticket office will disappear before the incoming rate of people is greater than 1000 per hour.	MR- hypothesis; C- seek approval; **T- part B; T- referral (direct);**
0:46:59.0	Mark	Okay. **Because students are served at a rate greater than**	This is said quietly, addressing no student in particular; he is **reading part B** of the problem either verbatim, or by rephrasing it in his own words.	**T- part B; T- referral (direct);** MT- rephrasing;

		that at which- The line decreases- Does it disappear before the students- What does the line begin, at when does the line begin to form, or begin to lengthen. That's a good question. Umm... Okay. Uh, wow! You can't do it without numbers, can you?	[While Mark is speaking aloud, Carina is doing her own work, Kam is shuffling through papers, Andrew is off-camera] Mark is indicating the need for actual numerical values in order to solve part B.	MR- difficulty (acknowledge); MR- need (numbers); MR- uncertainty; MR- hypothesis;

Other examples of codes in the task category are the initial condition of (0, 600), references made to parts A through F of the task, and other information about the rates of people arriving or leaving throughout the day as provided in the task. Coding for the task enabled me to see when students are referring to information provided in the task and when they are doing mathematical work in the explicit context of answering some part of the TLT. The context of students extending the task would, I thought, be an interesting avenue for risk taking and therefore needed to identify when students were referring to the task.

Another group of codes represent the mathematical concepts elicited by the TLT (MC for *mathematical concepts*). The code MC refers to general mathematical ideas discussed in students' brainstorming sessions, or that were developed in greater depth during student work. Examples of words coded as MC are algebra, geometry, area, etc. A transcript piece, provided below, illustrates the context and some examples in which mathematical concepts emerged during analysis of these data (references to mathematical concepts are bolded for easier identification).

Day 1B 0:49:12.4	Mark	When does yours cross the axis, like **cross each other**?	To Kam; asking when the lines they have drawn, $y_1 = 200x + 500$, $y_2 = -500x + 4000$, and $y_3 = 1000$ cross one another.	MR- seek understanding; **MC- intersection;** MT- y_1; MT- y_2; MT- y_3;
	Mark	Like what point do they- Do you know what point yours **crosses** it at?	[Rephrasing his question, camera angle is not on Mark well enough to see what he is gesturing to on his paper.]	MT- rephrase; MR- seek understanding; **MC- intersection;** MT- y_1; MT- y_3;
	Kam	Uhh, ten thirty?	[Kam hesitates when answering Mark with the time 10:30 am.]	MR- hypothesis; E- self-doubt; **MC- intersection;** MT- y_1; MT- y_3; T- 10:30am;
	Mark	Plus three, plus ten thirty, huh?	[Mumbling under his breath, his meaning cannot be understood; camera angle is too poor to gather physical clues]	
	Kam	I think it's exactly ten thirty.		MR- hypothesis; E- confidence (self); MT- exact;
		'Cause there's a **rise of one thousand**. And a **run** of five hours.	The rate of people getting in line from 8 am to 1 pm increases by 1000 over the duration of the five hours.	MR- justification; **MC- subtraction; MC- rise; MC- run;** T- interval C; MT- rate arriving; T- 1000;
		And so the **midpoint** would be I think just at	Because the initial rate of people entering the ticket office was 500 and the rate at 1 pm was 1500, and because it is a	MR- hypothesis; MR- justification; MT- rephrase;

		ten thirty. Just an increase of a thousand over five hours.	constant increase, Kam says that the midpoint of y_1 from 8 am to 1 pm is where the y-value would be 1000, the value halfway between 500 and 1500, and it happens at 10:30, the time halfway between 8 am and 1 pm. Interesting to note: this is where y_1 intersects y_2.	T- interval C; T- 10:30am; T- 1000; **MC- midpoint;**
0:49:47.3	Mark	Yeah, so why don't we like find the slo- I think that's what I'm gonna do.		MR- hypothesis; MR- plan;
		I'm gonna find the **slope** of this line right here.	Traces y_1 from 8 am to 1 pm with his pencil 	MR- plan; **MC- slope;** MT- y_1;

In the transcript provided above, students touch on general ideas such as the intersection of two lines, slope, rise, run, subtraction, and midpoint, all of which are examples of axial codes categorized as mathematical concepts. In total, 30 codes were developed that fit into the category of mathematical concepts (see Appendix B for complete list). Identification of the mathematical concepts students discuss is relevant to this study on risk taking as it helps to decipher student progress on the task and to identify major shifts in direction of student work, places I anticipated would be significant for student risk taking.

Student application of mathematical concepts led to specific mathematics elicited by the TLT (MT for *mathematics from task*), and included the specific equations, regions, etc. the students developed in their progress towards a solution. In order to identify student engagement in risk-taking activities, I needed to be able to know what specific mathematics they were developing and discussing so as to identify potentially dangerous mathematics that would require students to take mathematical risks. For a comprehensive list of student-developed equations and regions, see Appendix C.

Student actions, methods, and results of collaboration on the task comprised the (C) codes (C for *collaboration*). An excerpt from class is provided below to illustrate two codes that arose from these data: seeking comprehension, and seeking approval and occurred toward the end of Day 2A on the task, when Carina and Andrew are trying to develop the anti-derivative for y_6.

Day 2A 0:50:09.9	Andrew	And then you can find B because you know one of the points.	Referring to the constant, b, in their equation $y = \frac{200x^2}{2} + 500x + b$.	MR- propose; MR- extension; MR- justification; T- initial conditions; MC- constant; MT- y_6
0:50:14.3		Because- 'Cause **you know what I'm saying?**	Asks Carina if she is following his thought.	**C- seek comprehension**
0:50:17.9		'Cause we know it starts out at six hundred and zero.	Andrew says they can use the initial information that the total number of people in line at time zero is 600. His work shows that his anti-derivative equation is $y_6 = 100x^2 + 500x + 600$.	MR- justification; MT- anti-differentiation; T- initial condition;
0:50:21.6		So one of the points is, is, um… What d- Oh, zero, six hundred, **right?**	One of the points on y_6 is (0,600)	MT- work (graph); MR- conclusion; MT- rephrase; T- initial conditions; MC- coordinates; **C- seek approval;**

Other codes that emerged from these data and fit into the category of collaboration included student display of hesitation, comparing results or progress with others, acknowledging peers' work or progress, etc.

The conditions or circumstances dealing with student exhibition of emotional behavior (E for *emotion*) comprised another group of codes. This accounted for emotion displayed or admitted by students while they worked on the task. The following transcript from Day 2B briefly demonstrates the development of some codes for emotional behavior.

Day 2B 0:09:57.7	Andrew	I don't know. I gotta think about this better, **let's see**.		MR- uncertainty; **E- confidence;**
0:10:00.6	Carina	**Oh! Oh, okay.**	Throws her hands up in the air in excitement.	**E- excitement;**
0:10:02.5		**Let's try something.**		MR- hypothesis; **E- excitement;**
0:10:03.8	Andrew	You figured it out? **Okay.**		**E- support;**
0:10:05.0	Carina	**No that won't work**, never mind.		**E- self-doubt;**

Student expression of confidence, excitement, support, and self-doubt are evident in the above transcript. Examples of other emotional codes that emerged from data are expressions of belief, defeat, enjoyment, and satisfaction. Student expression of emotion was valuable for this study so as to provide information on the emotional aspect of student risk taking. A comprehensive list of codes in the E category is provided in Appendix B.

The final category of codes consisted of the conditions pertaining to and surrounding mathematical risk (MR for *mathematical risk*). In the following clip from Day 2B, Andrew has just finished explaining to Kam how he came up with the equation y_{7}. The clip illustrates emergence of some codes categorized as mathematical risk.

Day 2B 0:15:34.9	Andrew	**So that is how I got that equation.**	Concludes his elaboration of how he came up with y_7.	**MR- conclusion**; MT- rephrase; MT- y_7;
0:15:45.8		**I don't know** if that is right or not, but... (chuckles)	Admits that he is not certain that y_7 does represent the total number of people in line.	**MR- uncertainty**;
0:15:51.0	Kam	You know, **I don't know.**	Intonation implies that Kam honestly has no opinion on whether Andrew is right or wrong.	**MR- uncertainty;**
0:15:52.0	Andrew	Because **maybe you have to do two separate equations** because in reality you can't go below zero.	Proposes that maybe they need to find a piece-wise function because it's technically impossible for the number of people in line to be negative.	**MR- propose;** **MR- beginning;** MT- pieces; MT- length; **MR- meaning;** **MR- justification;** MC- reality; MC- negative; E- intuition
0:16:01.0		**So you have** to do one from eight to ten, and then one from 10:30 to-	The piece-wise function would include one function from 8 to 10, and another from 10 to 1 pm.	**MR- propose;** MT- elaboration; MT- pieces; T- interval (A-); T- interval (B);
0:16:06.5	Kam	**And then** one from ten to ten-thirty…	Recognizes that they'll need to account for the time from 10 to 10:30 by coming up with another function for it.	**MR- propose;** MT- elaboration; MT- pieces; E- support; T- interval (A+);
0:16:09.5	Andrew	Oh. **Using a different y intercept.**	Decides that they'll need a different y-intercept for the function from 10:30 to 1, but that everything else is the same because the input/output equations are not changing.	**MR- hypothesis;** MT- piece; MT- y-intercept;
0:16:14.5		You have to use a different y intercept, **because...**		MT- work aloud; MT- pieces; MT- y-intercept; **MR- justification;**
0:16:23.0		Ooohhh yea. **You'll have a different y intercept because in reality you can't go below zero.**	y_7 uses 600 for it's y-intercept because that was the number of people in line at time 0, but y_7 dictates that there must be negative people in line at 10:30, when in reality there cannot be negative	**MR- propose;** **MR- justification;** MT- pieces; MC- reality; **MR- meaning;**

			people in line, *and* the number of people in like at 10:30 must be zero.	MT- y-intercept;
0:16:29.7		**And so** you'll have to have a **different equation and a different y intercept.**	So the equation describing the number of people in line from 10:30 to 1 will be different from y_7 because it needs a different y-intercept.	MT- rephrase; **MR- propose;** **MR- justification;** MT- pieces; MT- y-intercept;
0:16:32.9		It will be about the **same parabola**, but it will be **a little bit higher up.**	Andrew claims that the function describing the people in line from 10:30 to 1 will only differ from y_7 in its height (it will be the same parabola translated up)	**MR- meaning;** MT- equivalence; **MR- propose;** MT- work (graph); MC- parabola; MC- translate (vertical);
0:16:39.2		And **it will account for that.**	The upwards translation of y_7 (i.e. different y-intercept) will account for there being 0 people in like at 10:30.	**MR- conclusion;** **MR- meaning;** MT- length (0); T- 10:30am; **MR- propose;**
0:16:40.3	Kam	**Okay.**	[an unconvincing tone]	**MR- believing game;**
0:16:42.2	Andrew	**Does that make sense?**		**MR- seek understanding;**

Justification, challenging, hypothesizing, counter-hypothesizing, student admittance of uncertainty, and seeking for meaning or understanding are some examples of the codes categorized as mathematical risk. The category of mathematical risk is pertinent to this study as I want to build theory to improve our understandings of the risks students take in learning mathematics. Therefore, I needed to identify the behaviors students exhibited during the learning process and determine which were integral to the learning process and which qualified as taking mathematical risk.

Chapter 6: Findings

6.1 Increase of Mathematical Activity

As students progressed in their work on the task, they seemed to be more engaged in all activity, particularly mathematical activity. While all video data were transcribed and given preliminary annotations, 38 total minutes of video were first annotated multiple times to provide rich detail and then coded. The 38 minutes of video came from three distinct pieces within the 8 ½ hours spent on the TLT. There were approximately 9 minutes taken from Day 1B, 10 minutes taken from Day 2A, and 19 minutes taken from Day 2B. Upon completion of axial coding, episode reports gave the frequency and duration of each axial code. I totaled the frequencies of each category of codes for the duration of the total 38 minutes that were annotated and coded. There seemed to be a general increase in student mathematical and risk-taking activity over the time spent on the TLT, and so first nine minutes of axially coded student work on the TLT (beginning approximately fifteen minutes into the task) were isolated and compared to the last nine minutes coded (occurring approximately 2 ½ hours into the task). Convenience dictated that 9 minute segments were compared; the first section of transcribed, annotated, and coded video was approximately 9 minutes long. It was logical, therefore, to compare the first 9 minutes to the last 9 minutes of video that were transcribed, annotated, and coded. The purpose for selecting and comparing the axial codes for these segments was to quantify student activity from the beginning of their work on the task and compare it with activity from the final stages of working on the TLT in the hope that direct comparison would provide information indicative of characteristics of student activity or the classroom environment that seemed particularly conducive to mathematical risk-taking.

Table 6.1.1 represents the direct numerical comparison of the first nine minutes to the last nine minutes of coded, annotated student work on the TLT.

Code	First nine minutes	Last nine minutes
Collaboration (C)	27	62
Emotion (E)	39	70
Mathematical Concepts (MC)	32	17
Mathematical Risks (MR)	63	110
Mathematics from Task (MT)	106	184
Speaker (S)	88	139
Task (T)	43	34

Table 6.1.1. Comparison of Code Frequencies.

6.1a Student Work on First versus Last Nine Minutes

The first nine minutes of coded, annotated student work began 15 minutes after students were given the task on Day 1. Students had already interpreted the TLT, creating personal representations for the provided information of the rates of people arriving and leaving the ticket office throughout the day. During the first nine minutes of coded, annotated work, students finished creating a graph of the rate of people arriving at the ticket office as a function of time, answering Part A of the TLT. Students developed the equations $y_1 = 200x + 500$ and $y_2 = -500x + 4000$ to represent the rates of people arriving at (or leaving) the ticket office from 8:00 am until 1:00 pm and from 1:00 until 4:00 pm respectively. Andrew put forth the idea to use area under the curve to answer Part B, which asked when the line of people at the ticket office would disappear and begin to form again.

Student work during the last 9 minutes of coded, annotated data comprised of Andrew and Carina working to develop $y_7 = 100x^2 - 500x + 600$ as the function representing the number of people in line from 8:00 am until 1:00 pm. During the last nine minutes they shared their work with Mark and Kam, and upon checking their equation with what they know to be the number of people in line at *x=2.5* (or 10:30 am) and *x=5* (or 1:00 pm) discover that y_7 yields values that are too high by 25 people. For the remainder of the nine minutes, Andrew and Carina worked to resolve the problem. Their work cumulated upon Andrew's proposal that "maybe you have to do two separate equations" (Day 2B, 15:52). At the end of the fifteen minutes, Andrew proposed the piece-wise function

$$y = \begin{cases} 100x^2 - 500x + 600 & 0 \le x < 2 \quad or \quad 8:00 - 10:00am \\ 0 & 2 \le x < 2.5 \quad or \quad 10:00 - 10:30am \\ 100x^2 - 500x + 625 & 2.5 \le x < 5 \quad or \quad 10:30am - 1:00pm \end{cases}$$

to represent the number of people in line at the ticket office from 8:00 am until 1:00 pm. Andrew later completed his piecewise function for the number of people in line at any time throughout the day.

6.1b Differences in Coding for First versus Last Nine Minutes

As is evidenced by Table 6.1, students increased in frequency in all but two categories of codes. The particular details of the codes that emerged from the first and last nine minutes provide compelling insight of the evolvement of student mathematical work and the development of personal meaning of mathematics.

6.1c Differences in Collaboration (C)

In particular, the collaborative efforts (C) put forth by the students during the first nine minutes consisted of 27 counts of five axial codes. The most common codes, or those present in five or more counts during the first nine minutes, were students rephrasing a previously-made statement (11 counts) and seeking approval from peers (9 counts). During the last nine minutes, a total of 52 counts of ten codes emerged from the data. The most common codes, or those present in five or more counts during the last nine minutes, were: confirming another's question (9 counts), rephrasing a previously-made statement (18 counts), seeking approval of one's own work from peers (8 counts), seeking comprehension of what a peer has stated (5 counts), and elaborating on a previously-stated idea (14 counts). Major differences between axial codes that emerged during the first versus the last nine minutes include a significant increase of student elaboration from 2 counts during the first nine minutes to 14 counts during the last nine minutes, the introduction of students confirming their peers' work with 9 counts during the last nine minutes versus 0 counts during the first nine minutes, and an increase in the frequency of students rephrasing their work from 11 to 18 counts. The increase of student behaviors such as rephrasing and elaboration indicate that students sought to develop a shared understanding of their mathematical work with their peers. Such behaviors contribute to collaborative problem solving as they enable students to communicate with one another based on the shared understanding they have developed. The increase of collaborative behaviors demonstrated by these students illustrates that when given ample time to work on the task without interruption in the form of direct instruction from the instructor, these students worked together to create mathematics and resolve issues inherent in the TLT.

6.1d Differences in Emotion (E)

The emotional behaviors (E) displayed by students during the first nine minutes constituted of 39 counts of eleven codes. The most prevalent, or those consisting of four or more counts, were: agreement with another's idea (4 counts), expression of doubt in a peer's idea (7 counts), expression of enjoyment in their work (4 counts), admission of self-doubt in one's own idea (4 counts), and support for peers and their mathematical competency (10 counts). In comparison, the last nine minutes of coded, annotated data included 70 counts of ten codes. The most prevalent, or those consisting of four or more counts, were: expression of belief in peers (12 counts), expression of confidence in self or peers (8 counts), display of excitement in their work (10 counts), employment of intuition (4 counts), and expression of support for one another and their mathematical competency (27 counts). Major differences between axial codes that emerged in the first versus last nine minutes include the decrease of student expressions of self-doubt during the last nine minutes of coded and annotated data, increase in student excitement in their mathematical work, and significant increase in student support and belief in one another. The overall effect is that after being engaged in the TLT for 2 ½ hours, students were more excited by their mathematical work, more encouraging of one another, and expressed confidence in their own and peers' mathematical competency.

6.1e Differences in Mathematical Concepts (MC)

The mathematical concepts (MC) discussed by students during the first nine minutes included 32 occurrences of fifteen codes. The most frequent, or those codes consisting of four or more counts included: the slope of a line (5 counts), subtraction or taking the difference between two values (4 counts), and use of a triangle (5 counts). During the last nine minutes of coded, annotated data, only 17 counts of six (MC) codes emerged. The most frequent included the use of negative values (5 counts) and the y-intercept (5 counts). Here the decrease from fifteen general mathematical concepts to six general concepts and the decrease in half of the total number of codes can be attributed to the fact that students initially survey the mathematical terrain and brainstorm possible methods to solve the TLT. After spending over two hours on the TLT, these students no longer needed to brainstorm solution methods; they were decided upon a solution method and therefore focused on the mathematical concepts involved in that method. In the last nine minutes of student work on the TLT students were focused on finding $\boldsymbol{y}_7$ and ensuring its accuracy in representing the number of people in line at the ticket office.

6.1f Differences in Mathematics from the Task (MT)

The greatest occurrences, i.e. six or more counts, of mathematics elicited by the task (MT) out of the 28 total codes that comprised 106 counts of MT in the first nine coded and annotated minutes of student work, were: use of area (6 counts), discussion of equations (6 counts), students cutting a graph or equation into smaller pieces (6 counts), use of the rate of people arriving at or leaving the ticket office (10 and 6 counts, respectively), student reference to their own work using graphs (7 counts), and use of $\boldsymbol{y}_1$ (7 counts). The last nine minutes showed the emergence of 184 counts of thirty-eight codes. Those with the greatest frequency, six or more occurrences, were: use of an anti-derivative (6 counts), use of area (9 counts), students checking their work to ensure accuracy (14 counts), students discussing the line length (19 counts), use of cutting a graph or equation into pieces (11 counts), region B (6 counts, Appendix B), seeking clarification of peers' statement (6 counts), students working aloud (14 counts), students performing calculations (14 counts), student use of a graphing calculator (14 counts), students referring to a graph they have created (7 counts), y_6 (6 counts), and y_7 (12 counts). In general, students' mathematical work evolved so that students were developing equations for the number of people in line as opposed to the rates of people getting into or out of line at the ticket office. Students increased in calculator and graph usage and worked diligently to check their work so as to provide answers to the TLT that were correct. Students are also referring to specific equations in high frequency, are doing calculations, and performing higher-level mathematics such as taking the anti-derivative.

6.1g Differences in Task References (T)

Student references to the task yielded 43 total counts of fourteen (T) codes. The most common individual components, i.e. five or more counts, were: direct and indirect references made by students to various parts of the TLT (6 and 5 counts, respectively), the time interval from 8:00 am until 1:00 pm (6 counts), and the rate of people leaving the ticket office, 1000 (5 counts). The last nine minutes compared by producing 34 total counts of twelve codes. The most frequent were: 10:30 am (7 counts), and reference to the initial condition that at 8:00 am 600 people were already in line (5 counts). During the last nine minutes, there was not one single reference, direct or indirect, to any part of the TLT. Student work had moved away from the specific questions asked in the TLT, and students were extending their ideas beyond the scope of what the TLT required. In particular, students were developing equations for the number of people in line, but the task only required students to create a graph of the length of the line as a function of time (Part F). Students interpreted the meaning of language in the task and embedded those meanings in their mathematical work; they moved beyond the language of the task itself and their discourse centered on language that reflects or refers to their mathematical work and problem solving solutions as students focus on increasingly difficult mathematics.

6.1h Differences in Mathematical Risks (MR)

The behaviors associated with mathematical risk (MR) displayed by students during the first nine minutes of coded, annotated work on the TLT included 63 counts of twelve codes. The most frequent, or those axial codes with six or more counts, included: hypothesizing (17 counts), statements of justification (9 counts), seeking understanding for oneself (6 counts), and admittance of uncertainty (7 counts). In comparison, 110 counts of fifteen codes emerged from the last nine minutes of coded, annotated data. The most frequent were: use of a concluding statement (9 counts), hypothesizing (14 counts), admittance of increased understanding (6 counts), justification (11 counts), actions based on personal meaning of student work (25 counts), change in perspective (6 counts), proposing a new idea (8 counts), and admittance of uncertainty (15 counts). Notable changes from the first to the last nine minutes of coded, annotated student work include an increase in students proposing new ideas from 4 to 8 counts and an increase in student admittance of uncertainty from 7 to 15 counts. Such changes indicate that although students claim to be experiencing uncertainty with greater frequency, they continue to press forward in their work by proposing new mathematical ideas that lead them to develop significant understanding of anti-differentiation.

The most significant difference in the count for an axial code from the first to the last nine minutes of coded, annotated work was the appearance of student actions based on personal meaning of their mathematical work; the count for the first nine minutes was 0, compared to 25 during the last nine minutes. This demonstrates that as these students progressed in their work on the TLT, they became more focused on the meaning of the mathematics in which they were engaged; students grounded their work in their understandings of what they were doing. Counts of other axial codes reflective of mathematical risk, such as hypothesizing, justifying, and providing concluding statements proved consistent when the first nine minutes of coded, annotated student work were compared to the last nine minutes. The cumulative effect of the increase of students grounding their work in mathematical meaning, admittance of uncertainty, and proposing new ideas is that of a general increase of student risk taking and development of personal understanding and mathematical meaning. These students chose to engage in difficult mathematics and press forward when faced with uncertainty; through grounding their mathematical work in real-life meaning, they were able to make sense of the anti-differentiation and use it to develop one piece-wise function to describe the number of people in line at the ticket office at any time throughout the day.

6.1i Summary of Student Increase in Activity

Sullivan, Tobias, and McDonough (2002) and Guthrie (1997) argue that risk is associated with emotion. In these qualitative data, the increase in mathematical risk and emotion over time as students worked on the TLT seems to demonstrate a positive, qualitative relationship between risk and emotion and suggests that for these students, risk is associated with emotion. Therefore an increase in emotional attachment to student work would seem to provide increased opportunity for students to take risks in the classroom.

As students were given time to work on the task, they became more emotionally involved with their work and progress toward a solution, their collaborative efforts increased, and they were more inclined to engage in significant mathematical activities grounded in personal meaning.. The increase of collaborative efforts is evidence that in the context of the research classroom, students increased their mathematical productivity and risk-taking behaviors as they were given time to delve deeply into the task. In order for these students to take mathematical risk, they were first engaged in mathematical activity in a context that gave them the opportunity to exercise agency in mathematical problem-solving through working on a task designed to elicit difficult mathematics.

6.2 Student Enjoyment in Uncertainty

Throughout student work on the task, they displayed enjoyment in their work amidst acknowledged uncertainty in their progress towards a solution and the validity of proposed ideas. In the following transcript piece, Andrew and Carina are packing up their bags at the end of the first day of work on the TLT. Immediately prior to the excerpt, Andrew had suggested to Kam that they find an equation for the number of people in line based on the anti-differentiation of y_1 and y_2, an idea which Kam said he could not "see how to do…in [a] math way" (Day 1B, 53:29.5). Andrew responded by saying, "I'll just forget about it, I guess," (Day 1B, 54:07) although he eventually returns to it and expounds upon it (Day 2A, 49:09), leading to the unique and correct piece-wise function discussed earlier. In the transcript below, Andrew and Kam are trying to make sense of finding an equation that represents the number of people in line at the ticket office. Immediately after admitting he did not know what he was doing and that he intended to forget about his idea, Andrew declared that working on the TLT was fun:

Time	**Speaker**	**Verbatim**	**Annotations**	**Code(s)**

Day 1B 0:53:50.2	Andrew	I'm just- Like, how would we *come up with the actual equation* for the- You know what I mean?	Traces the line $y_3 = 1000$ with his pencil, implying that the final equation using integration should account for both the incoming *and* the outgoing rates of people in the ticket office.	MT- rephrase; MT- seek method; MR- seek understanding; C- seek comprehension; MR- plan; MT- rate (arriving); MT- rate (leaving); MT- y_3; *MT- integration;*
0:53:56.3	Kam	Oh yeah.	Kam realizes he had not been taking y_3 into account with the plan he had shared.	MR- perspective; E- support; MR- increased understanding;
		Subtracting the one thousand.	Recognizes that they do need to account for both the incoming and outgoing rates of people when finding a general equation for the number of people in line.	MR- rephrase; *MT- rate (leaving);* MT- y_3; *MC- subtraction;* *MT- rate (arriving);* *MT- combine;*
0:54:00.2	Andrew	**I don't know. I don't know.**	Stating either that he does not know how to actually find the anti-derivative and/or take into account the rates of people going both in and out of the ticket office.	**MR- uncertain**; MT- integration; MT- rate (leaving); MT- rate (arriving);
0:54:03.7	Kam	**I don't know either.**	Stating that he, too, does not know how to find the anti-derivative.	**MR- uncertain**; *MT- integration*; MT- rate (leaving); MT- rate (arriving);

				E- support;
0:54:07.5	Andrew	**I'll forget about it** I guess.	Decides to throw away his idea of integrating and work from a different angle.	**E- dismiss**
In the background, the instructor is heard discussing homework and assessments as the class period is ending.				
0:54:15.5	Andrew	**This is kind of fun.**	Referring to the Ticket Line Task and/or the initial work they have been doing on it.	**E- enjoyment;**
0:54:16.7	Carina	**Yeah it is.**	Agrees with Andrew that their work is fun.	**E- enjoyment;**

In the transcript excerpt provided above, Andrew and Kam are coming to the end of their struggle with Andrew's idea of taking the anti-derivative of y_1. They both admit that they cannot find the anti-derivative, and Andrew declares his intent to forget his idea and work on the TLT using a different approach. Immediately after doing so, Andrew turns to Carina to express that he is having fun working on the TLT. This example provides evidence that students are capable of working diligently on difficult mathematics like anti-differentiation even when they do not know what future direction to take to progress towards a solution. This piece also demonstrates that students in the context environment are capable of collaborating with peers to the extent that their focus is narrowed on their mathematical work; they willingly lose themselves in their work, and in the process of doing so find enjoyment in mathematics. Given that risk is connected to student emotions, it follows that as students are able to experience positive emotions while grappling with uncertainty, they willingly engage in risk-taking activities leading toward potential solutions of the task.

6.3 Mathematical Argumentation

As students worked on the task, they continually engaged in hypothesizing, justifying, and challenging or counter-hypothesizing as they strove for mathematical understanding. The codes for hypothesizing, justifying, challenging, and counter-hypothesizing were persistently located in close proximity, and the collective effect is mathematical argumentation (NCTM 2000; Stein, 2001; Walter, Rosenlof, & Gerson, 2008; Yackel & Cobb, 1996). Students dispute various mathematical hypotheses, providing justification and reasoning for each, and they pursue that which seems most likely to lead to understanding. Mathematical argumentation is directed toward building consensus in meaning (Walter & Johnson, 2007), and the result of such cases was usually an increase in understanding. For example, during the end of Day 2A (54:22) Carina puts forth an idea to answer Part D of the task, which asked students to find when the line at the ticket office would finally disappear. She hypothesizes that the line of people at the ticket office will actually never disappear, reasoning that people are still arriving at the ticket office and will contribute to the line before it can empty. Andrew disagrees with her and claims that the line actually does disappear around the time when y_1 and y_3 intersect (i.e. 10:30 am), as seen in the following example:

Time	**Speaker**	**Verbatim**	**Annotations**	**Code(s)**
Day 2A 0:54:22.9	Carina	So the question when it says, "does the line disappear", **so it never disappears**.	Referring to part d of the question: "About when does the line finally disappear," and claims that the line never will disappear…	T- part D; T- referral (direct); **MR- hypothesis**; MT- length(0); MT- rephrase;
0:54:27.6		**Cause it's still coming in**.	…because people will always be entering the ticket office.	MT- rate (arriving); **MR- justification;**

0:54:29.0	Andrew	Um, It- I think we figured out **it doe**s.	Informs Carina that the line *will* disappear at some point.	MR- beginning; **MR- counter-hypothesis;**
0:54:31.7		**Because right here it's going down down down**, **I think the line does disappear** before it gets to this point, but **then it just starts to grow again**, after that point.	Explains that as time approaches the intersection of y_1 and y_3 (i.e. 10:30 am), the line gets smaller and smaller and smaller [with his arms in the air he draws an upward-facing parabola as it approaches its vertex from the decreasing side] because the rate of people being served is greater than the rate of people getting in line. After the point of intersection, the line starts to grow again… [Carina is staring at Andrew, looking very unconvinced].	**MR- justification; MR- hypothesis;** MT- body; MT- length;
0:54:42.3		**Because, cause this whole point, the line's getting smaller, smaller, smaller**. It all started at 600.	Gestures to interval A, saying that during that time the line, which started having 600 people in it, gets smaller and smaller.	**MR- justification;** MT- rate(compare); MT- length (decrease); T- initial condition; MT- body ; T- interval A ;
0:54:49.8		**So for this whole thing it's getting smaller and smaller and smaller and smaller…**	Restates what's going on and shows an interval between his hands that gets smaller and smaller, demonstrating the length of the line.	MT- rephrase; **MR- justification;** T- interval A; MT- length (decrease); MT- body;
0:54:52.8		But then finally, and then	At "finally" he uses his pencil to point to the intersection of y_1 and	**MR- conclusion;** MT- length (0);

		there's- and **then the line's gone-**	y_3. To demonstrate the line being gone, he makes a motion similar to an umpire in baseball when calling people "safe."	MT- rephrase; MC- intersection; T- 10:30 am; MR- conclusion;
0:54:56.1	Carina	**Oh, okay**.	The unconvinced looks have ceased, Carina seems completely at peace with Andrew's explanation.	**MR- increased understanding;**
0:54:57.4	Andrew	**And then it goes bigger, bigger, bigger, bigger…**	Uses his hands once again to demonstrate that after 10:30 am the line will grow again	**MR- hypothesis;** MT- length (increase); MT- body;
0:55:00.3		**I think**.	(chuckles)	**MR- uncertainty;**

In this example we see Carina hypothesize that the number of people in line will never reach zero, providing justification that the line will not disappear because people are still arriving at the ticket office and getting into line. Andrew then challenges her idea with a counter-hypothesis that the line does in fact disappear before 10:30 am, reasoning that although people are still arriving at the ticket office, because the rate of people being served is greater than the rate of people arriving between 8 and 10:30 am the length of the line will decrease from 8:00 until disappearing sometime before 10:30 am, and then after 10:30 am the line will begin to increase in length.

Students expressed reasoning and offered justifications for mathematical inferences throughout their work on the TLT. Such interactions of hypothesizing, counter-hypothesizing, and providing justification generally resulted in some observable acknowledgement of increased understanding on behalf of one or more of the participants. Mathematical argumentation proves to be risky as students bridge the differences between their hypotheses, justifications, and mathematical inferences to those provided by others. In the above transcript, Andrew admitted lack of certainty in the correctness of his idea (0:55:00.3), yet still found the endeavor of challenging Carina to be worthwhile. Mathematical argumentation, therefore, can be characterized as a risk students take in negotiating mathematical meaning.

6.4 Intellectual Adventuring

An important distinction must be made between hypothesizing and proposing a new idea. Proposing a new idea is defined here to be the introduction of a significant mathematical idea that had not previously been discussed, such that its implementation would alter the direction of students' mathematical course. Hypothesizing is the articulation of a student's brainstorm of thoughts about various mathematical routes that could potentially help students progress towards a solution. At a few particular places in student work on the TLT, I saw students' mathematical progress shift due to a newly proposed hypothesis. These shifts in students' progress were typically categorized by student acknowledgement of uncertainty regarding the hypothesis' potential to help students progress or the accuracy of the speaker's idea. I call such experiences "intellectual adventuring" as students are exploring new mathematical pathways intellectually, and because they saw themselves as adventurers in the mathematical world. The following excerpt illustrates students viewing themselves as adventurers and is taken from Day 1B as students have decided it is possible to create equations to represent the rates of people arriving and being served by the ticket office throughout business hours, and that their graph representing the rates of people arriving or being served throughout the day is accurate.

Time	Speaker	Verbatim	Annotations	Code(s)
Day 1B 0:48:30.7	Kam	**Oh adventurous**.		E- enjoyment; **E- adventure;**

0:48:34.3	Mark	**In the mathematical world.**	Adding onto Kam's statement that their work is *mathematically* adventurous	E- enjoyment; **E- adventure;**

This excerpt shows that Kam and Mark viewed their mathematical work on the TLT as stimulating, challenging, and ultimately adventuring.

An example of student behavior characterized as intellectual adventuring is provided below. Andrew has just proposed $y_7 = 100x^2 - 500x + 600$ and its corresponding graphs (Figure 6.4.1) to represent the number of people in line from 8:00 am to 1:00 pm.

Time	Speaker	Verbatim	Annotations	Code(s)
Day 2B 0:00:24.2	Andrew	I just figured something cool out.	States that the integral work he has done is "cool," important, and worth mentioning.	E- Self-belief; E- enjoyment;
0:00:25.7		**I don't know if it's right**, but-	Admits he does not know whether his work is right or wrong.	**MR- uncertainty;**
0:00:27.4		I think if we- **I think this would be about the equation** for the number of students, from, this is gonna be eight, this is gonna be one, right?	Andrew points to the equation $y_7 = 100x^2 - 500x + 600$ in his work and shows Kam the graph he had created representing the total number of students in line from 8 am to 1 pm. He explains that y_7 starts at 8 am, and points to the starting point of his graph, then points to the end of his graph, at 1 pm.	**MR- propose;** MT- elaborate; **MT- y_7;** MT- length; T- interval C; MT- work(graph); T- 8am; T- 1pm; C- seek comprehension;

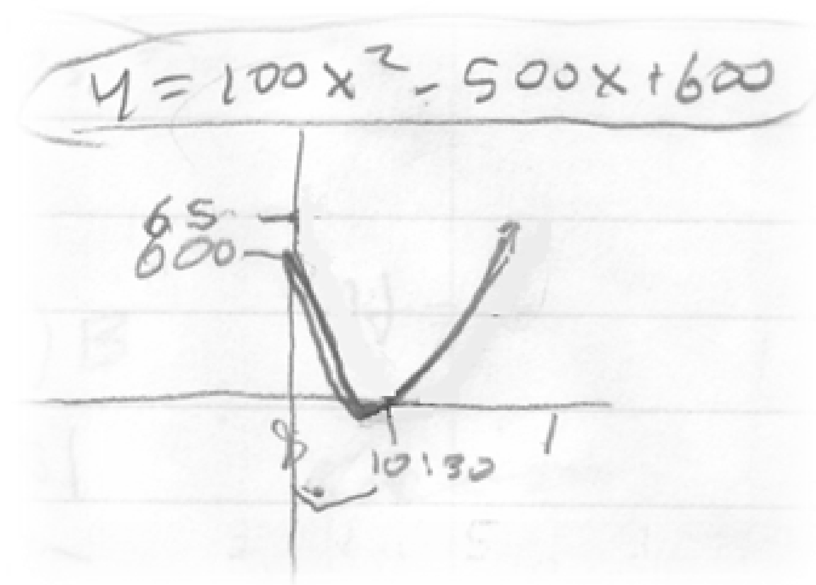


Figure 6.4.1. Andrew's Graph of y_7.

In the transcript provided above Andrew has just proposed y_7 as the function equation that represents the number of people in line from 8:00 am until 1:00 pm. This qualifies as intellectual adventuring because Andrew is admitting uncertainty in the validity and potential of his idea to help him progress towards a solution, but he nonetheless introduces it to his peers as an idea worthy of pursuit. Because of Andrew's confidence in his work, he continues to follow this path and it provides him with the foundation of understanding that it is necessary to develop the piece-wise function representing the number of people in line at the ticket office any time throughout the day.

Intellectual adventuring qualifies as an emotional risk that students take because they are subjecting their own mathematical work and ideas to their peers for public scrutiny. More importantly, intellectual adventuring is a mathematical risk as students are traveling unknown mathematical pathways. Each student must decide how to handle the difficult mathematics that could prove to be dangerous ground. In developing one piecewise function to represent the number of people in line at the ticket office during business hours, Andrew needed to decide how to handle anti-differentiation of the corners or cusps of the rate functions (i.e. 10:00 am, 10:30 am, 1:00 pm, 3:52 pm), as such corners cannot be anti-differentiated. Essentially, Andrew used a piece-wise function to describe the various time periods that existed throughout the day (8:00 – 10:00 am, 10:00 – 10:30 am, 10:30 am – 1:00 pm, 1:00 – 3:52 pm, and 3:52 – 4:00 pm) and took the anti-derivative of the composite rate function for each time period. For example, Andrew took the anti-derivative of $y = 200x - 500$ (the composite rate function from 8:00 until 10:00 am), which he found to be $y_7 = 100x^2 - 500x + 600$, and said that y_7 represented the number of people in line from 8:00 until 10:00 am. Andrew checked y_7 at 10:00, found that it equaled 0 (what he had found when using area under the curve to know the number of people in line at 10:00), and reconciled that if $y_7 = 0$ at 10:00 am, and $y = 0$ from 10:00 until 10:30 am, he could just define y_7 to be the function through and including $x = 2$, or 10:00 am (Figure 6.4.2)

$$y = \begin{cases} 100x^2 - 500x + 600 & 0 \le x < 2 \quad or \quad 8:00 - 10:00am \\ 0 & 2 \le x < 2.5 \quad or \quad 10:00 - 10:30am \\ 100x^2 - 500x + 625 & 2.5 \le x < 5 \quad or \quad 10:30am - 1:00pm \\ -250x^2 + 3000x - 8125 & 5 \le x < 7.87 \quad or \quad 1:00 - 3:52pm \\ 0 & 7.87 \le x < 8 \quad or \quad 3:52 - 4:00pm \end{cases}$$

Figure 6.4.2. Andrew’s Piece-Wise Function.

Although Andrew proposed that y_7 would represent the number of people in line at any time from 8:00 am until 1:00 pm, and it actually only represented the number of people in line from 8:00 until 10:00 am, dealing with the errors that resulted from checking y_7 at 10:30 and 1:00, which yielded -25 and 600 people respectively, and the actual 0 and 625 people in line at 10:30 am and 1:00 pm provided Andrew and Carina the opportunity to deal with changing composite rates of people arriving at the ticket office and led them to develop the piecewise function. Intellectual adventuring, therefore, is a mathematical risk these students took in the mathematics classroom.

6.5 Students Recognize Benefits of Risk

Students not only recognized that they were engaging in activities involving uncertain outcomes, but also acknowledged such benefits as increased understanding and additional support for mathematical claims. In the transcript excerpt provided below, Andrew tells his peers that his motivation for trying to use integration (anti-differentiation) to solve the TLT is because it will help him "learn more." Kam agrees with Andrew, stating that trying it will help them understand, and Carina adds that doing so will help support the claims they have made using area under the curve to find the number of people in line at 10:00 and 10:30 am.

Time	**Speaker**	**Verbatim**	**Annotations**	**Code(s)**
Day 2B 0:01:48.5	Andrew	We're gonna do it just to kind of try to figure out. **Just to learn more.**	Andrew explains that he and Carina want to continue on their integrating path to see where it goes and what they learn.	MR- beginning; **MR- benefit;** MR- plan;
0:01:51.6	Kam	Yeah. 'Cause it'll **help us understand.**		E- agree; E- support; **MR- benefit;** MR- seek understanding;
0:01:52.2	Carina	**Support our statement.**	[kind of under her breath]	E- support; **MR- benefit;**

Atkinson (1957) and Clifford (1991) described the positive, linear relationship between an increase in success in the classroom and an increase in the amount of risks students take. The students observed in this study expressed in their pre-semester perspectives surveys that they viewed persistence for mathematical understanding as something positive, important, and even a definition of success (Appendix A). Here we see evidence that these in their practice at the micro-level halfway through the semester, these students viewed increased understanding and additional support for their mathematical hypothesis as important, even a measure of success, and therefore evidence that for these students there *is* a positive relationship between their view of increased success in the mathematics classroom and their willingness to take risks. It is important to note that although a linear relationship exists between student success and risk-taking, student learning and growth of understanding (Pirie & Kieren, T., 1994) travels a non-linear path through the various layers of understanding.

6.6 Purpose and Meaning through Uncertainty

As data were processed multiple times, I became aware of a general occurrence where students were developing purpose and meaning for mathematics through their experiences handling uncertainty. In developing a graph and equation to represent the number of people in line from 8:00 am until 1:00 pm, Andrew admitted multiple times that he was unsure whether what he was doing was correct or if it was even leading him towards a solution. In the excerpt provided below, Andrew claims that following through with his work has led him to develop personal meaning for anti-differentiation, even though he is still unsure whether it was correct. In fact, soon afterwards he and Carina discover that y_7 only represents the number of people in line until 10:00 am, not 1:00 pm as he originally claimed. Although y_7 was only partially correct, its development provided opportunity for Andrew to explore and discover applicable meaning for anti-differentiation.

Time	Speaker	Verbatim	Annotations	Code(s)
Day 2B 0:10:55.3	Andrew	So that makes sense...	Looking at the graph of $y_7 = 100x^2 - 500x + 600$, he concludes that it could reasonably represent the total number of people in line	MR- meaning; MT- work (graph); MT- y_7;
0:10:57.1		…but I don't know if it's right.	Andrew is claiming that he is not 100% confident in y_7 as the most appropriate function to represent the total number of people in line.	MR- uncertainty; E- self-doubt; MT- y_7;

Throughout student work on the TLT, they displayed an eagerness to develop personal meaning and understanding for the mathematics in which they were engaged. Upon facing potential mathematical obstacles such as instances where anti-differentiation is not possible, the students reconciled the obstacle by checking the piece-wise function at each of the boundary changes and ensuring that each function yielded equivalent numbers for the amount of people in line at 10:00, 10:30, 1:00, and 3:52. Students developed greater understanding for anti-differentiation through willing engagement in such risks as mathematical argumentation and intellectual adventuring. They claimed that taking such risks would further their understanding and help provide support for earlier mathematical work on the TLT.

Chapter 7: Conclusions and Discussion

These data and analysis demonstrated that students willingly take risks in the mathematics classroom in various contexts which contribute to their mathematical understanding and, ultimately, to their mathematical success. Findings showed that students increased in activity as they worked on the task, engaged in mathematical argumentation and intellectual adventuring, developed purpose and meaning for the mathematics they were dealing with, and enjoyed their work, ultimately viewing themselves as mathematically successful.

As the findings were grounded in data, the context of the classroom is vital to conclusions I make. It is important to note that students working on the TLT were enrolled in a learner-centered classroom that focused on student development of personal understanding of the meanings and purposes of mathematics. Students exercised personal agency in choosing the degree in which they participated and were engaged in mathematics. They also chose their own learning pathways; there was no teacher dictating the subsequent steps to follow or correcting students in the slightest mistakes. Students were permitted to take risks in anticipation that doing so would prove beneficial. Students were also given challenging tasks with unknown outcomes that pushed them outside their mathematical comfort level. They had sufficient time to grapple with the task and the involved mathematics and experience the result of doing so. The task was also accessible to students as it provided multiple avenues towards a solution in allowing them to use a variety of methods to solve it, as seen in Kam and Andrew's varied approaches to discovering when the line at the ticket office would disappear.

In such a context, students displayed eagerness and excitement in engaging in significant mathematical activity. They exhibited "powers of intellectual passion" and "tendencies in action" to persevere past merely attaining correct answers to the growth and building of conceptual understandings of integration (Walter, Hart, et al., 2009). Students acknowledged benefits of taking risks. They recognized and admitted uncertainty, and claimed that working through it would help them by providing increased understanding. Lastly, students achieved success as defined by themselves, the instructor, and the researcher in building rich understandings of the meanings and purposes of mathematics.

7.1 Contextual Risk Theory: CRT

Contextual Risk Theory (CRT) asserts that students take risks when they engage in mathematical argumentation and intellectual adventuring, that in grappling with uncertainty students find enjoyment in mathematics and develop understanding of the purposes and meanings of significant mathematics, that students view themselves as adventurers and recognize benefits of taking risks, and that students employ personal agency in becoming more involved emotionally and mathematically when presented with the opportunity to work on appropriately challenging tasks placing high demand on their problem solving and reasoning skills.

7.2 Mathematical Argumentation

Analysis of data showed that students will become engaged in their mathematical work and become problem posers and participants in mathematical discourse. They participate in hypothesizing and brainstorming directions for their work, they claim validity of their work and voluntarily or at the request of peers justify their reasoning for their work's validity, and display comfort and confidence in challenging one another's work. Engaging in mathematical argumentation is characterized as taking a risk because students are acting on self-belief and conviction that their mathematical ideas are valid and valuable while challenging those ideas they disagree with. Demanding justification from peers or challenging others' ideas contributes to risk-taking activity as it sends a definite and unmistakable signal to others that the one challenging or demanding justification is not convinced of the certainty of another's idea. It is important to note that throughout student work on the TLT, it was students' mathematical hypotheses that were challenged or embraced. Further research could provide insight to how one could develop such a classroom and to identify key components of such a classroom. The classroom in this study sustained mathematical risk taking activities as its members determined the mathematical discourse and provided students with an environment of safety, comfort, and respect. In this study, students were given a task that challenged them to become problem posers and to wrestle with mathematical uncertainty. The instructors demonstrated the need for justification and reasoning throughout student work, and provided students with opportunities to participate in mathematical argumentation.

7.3 Intellectual Adventuring

Students displayed an eager willingness to explore mathematics inherent in the TLT. At a number of occasions in their collaborative work on the task, the students reached mathematical crossroads where a new, unexplored idea was proposed and compared to the familiar work students were already involved in. Students admitted the uncertainty involved in the potential outcome and validity of newly proposed ideas, yet in choosing the path less traveled, they recognized the benefit of increased understanding that would result from doing so.

7.4 Increase of Mathematical and Risk-taking Activity

The more time students were involved in working on the TLT, the more active they became collaboratively, emotionally, and mathematically. Students participated in verbal problem solving and mathematical discourse at an increased frequency and displayed greater emotional attachment to the mathematics involved through expressions of enjoyment, doubt, and confidence. Student actions evolved over their work in the task to be more riveted and focused on significant and advanced mathematics such as anti-differentiation, and student actions were centered on the personal meaning and understanding they created for the mathematics in which they were engaged.

7.5 Purpose and Meaning through Uncertainty

In grappling with uncertainty, students came to develop a rich understanding of the meanings and purposes of specific mathematical concepts. They understood the application of and developed real-life meaning for anti-differentiation. Risk-taking is therefore evident as students deal with uncertain outcomes of their work, and through pursuit of their work on the task they achieved success through growth in mathematical understanding of anti-differentiation.

7.6 Student Enjoyment

Students expressed enjoyment in participating in mathematical problem solving throughout their work on the TLT. They declared that they were having fun, even when their ideas were proven wrong and they did not know what direction they needed to take to progress towards a solution. If students enjoy their work, regardless of the outcome, certainly they will be more likely to engage in mathematical risk and the benefits will greatly outnumber the costs of doing so. Therefore, student enjoyment indicates a likelihood of risk-taking.

7.7 Success

Based on student statements that mathematical understanding is the standard for success, student persistence and eventual understanding of anti-differentiation as a method to use the equation for the rate of people getting in line at the ticket office to create an equation for the number of people in line at the ticket office is evidence that these students were, by their own measure, successful. They expressed excitement and enthusiasm in their solutions and the effort they put into the task. After Andrew and Carina decided what the first three pieces of the piece-wise function would be, Andrew exclaimed, “Ba-bam! We figured it out!”

On the pre-semester perspectives survey, Andrew reported that the optimum classroom environment for learning mathematics is a procedure-oriented classroom where the instructor provides a basic introduction to the topic, gives a real-world explanation of the topic, then expects students to practice with homework and quizzes. Andrew claimed that an excellent mathematics teacher is one who will spend time solving multiple problems using a variety of approaches for the class. Andrew complained that difficult mathematics problems “only challenge[s] intelligence and not math competency” and wanted “many, many problems” to do that were only moderately difficult as opposed to several problems of significant difficulty. The honors calculus class was set up fundamentally different from Andrew’s “optimum classroom” in several ways. The instructors did not use a traditional format where student work is comprised of practice and application; instead the instructors used challenging tasks to elicit mathematics and encouraged students to use personal agency in problem solving.

Throughout his work on the TLT, Andrew's behavior contradicted his pre-semester statements and indicated a shift in his views on what a mathematics classroom and mathematics teachers ought to look like. At the end of the first day working on the TLT, and immediately after stating that he did not know how to fine an anti-derivative, Andrew said of his work, "this is fun" (Day 1B, 54:15). He claimed to be enjoying his experience although he was dealing with mathematical uncertainty accompanying significant mathematics. When Andrew and Carina realized how to find an anti-derivative, he declared his intent to apply anti-differentiation "just to learn more" (Day 2B, 1:48). Andrew viewed personal increase in understanding as important and through exploration of anti-differentiation he not only developed a piece-wise function to represent the number of people in line at any time throughout the day, but also saw his work as "cool" and worthwhile (Day 2B, 0:24).

Andrew's enacted perception of success is, therefore, fundamentally different from his pre-semester proclaimed view of success. Andrew demonstrated an eagerness to engage in difficult mathematics solely for the intent of increased understanding. In using anti-differentiation Andrew displayed He displayed excitement and pride in his own mathematical competency through working on one very difficult task; his work showed that in working on one task, students can develop rich understanding of significant mathematics.

The students were delighted with the work they had done and thrilled with the understanding they had developed. In overcoming obstacles that existed due to the elements of uncertainty with which students were grappling, students viewed themselves as mathematically competent and ultimately successful. Uncertainty, therefore, plays a vital role in students achieving success.

As students took risks by participating in mathematical problem solving, intellectual adventuring, and mathematical argumentation, they indicated increased understanding through verbal admittance or using words, gestures, or behaviors indicative of increased understanding such as a prolonged “oooh” with falling intonation and facial expressions implying that the speaker has experienced growth in understanding. Students began the task with the provided information and the intuitive knowledge of anti-differentiation, or finite integration as area under the curve. As they began to grapple with the task, the concept of integration as area under the curve was developed and strengthened as they compared their results using area under the curve with other methods students accepted as legitimate. Andrew and Carina in particular ultimately invented a piece-wise function to represent the number of people in line at the ticket office throughout the day.

Chapter 8: Implications and Future Research

CRT maintains that as students participate in mathematical argumentation and intellectual adventuring, they are taking risks. In taking risks, students view mathematics as enjoyable and view themselves as mathematical adventurers. Students recognize the benefits of mathematical risk taking and employ personal agency in choosing to take mathematical risks. As students take the opportunity to work on challenging tasks that place demand on their problem solving and reasoning skills, they develop understanding of the purposes and meanings of significant mathematics.

Implications of CRT include the need for challenging tasks that introduce uncertainty and elicit problem posing by students, the need for teachers to foster classroom environments conducive to risk-taking among students, the importance for teachers to provide their students with adequate time to work on tasks, and teachers who are willing to take mathematical and personal risks to provide their students with more effective learning opportunities.

8.1 Appropriately Challenging Tasks

CRT emerged from a context in which students were given a challenging task used to elicit mathematics, rather than as a practice problem, that included multiple avenues towards a solution. We saw that the use of the task provided students with a reason to take mathematical risks that were productive to their learning. Students will be more likely to take risks that lead them to build strong mathematical understanding if teachers place demands on their students’ problem-solving and reasoning skills by giving students challenging tasks that elicit mathematics instead of telling students precisely what to do and having them copy or repeat the teacher’s example. Teachers can provide students with the opportunity to take mathematical risks by introducing uncertainty into the classroom and allowing students to stretch outside their usual comfort zones. This research demonstrates that in grappling with uncertainty in the mathematics classroom, students negotiate and develop personal understanding for the meanings and purposes of mathematics.

8.2 Classroom Environments

The context from which CRT emerged was a student-centered classroom where students exercised personal agency in mathematical problem solving. One implication of CRT is the nature of classroom environments. When teachers put a high priority on inquiry, agency, and student problem-posing in mathematical learning, students may be more likely to take risks resulting in strong mathematics learning. Teachers can provide their students with experiences suited to their needs rather than focusing on course objectives, standardized tests, or the need to maintain a fast pace despite student misconceptions. As teachers expect and encourage intellectual risk-taking among their students and strive to develop supportive communities of learning, students feel companionship and safety among their peers (NCTM, 2007). A classroom conducive to risk will be comprised of individuals that value discussion of significant mathematics and an educator that demonstrates appropriate mathematical discussion and helps students develop proper habits of discussion, including the distinction between focusing on students' mathematical work versus the student offering their work for discussion. These actions could help students feel confident hypothesizing among peers and increase the benefits of intellectual risk while minimizing the costs of emotional risk. The findings of this study demonstrate the positive implications for learning when teachers develop habits that encourage students to exercise personal agency in mathematical problem solving, rather than simply telling students pertinent information and expecting them to develop understanding. The students in this study demonstrate the ability to make their own decisions about the mathematics that contribute to personal development of rich mathematical knowledge.

8.3 Adequate Working Time

In this research Andrew made a significant break-through leading to quick resolution of the latter portion of the task only after working on the task for over two hours. Had the instructor been hasty in rushing Andrew along or in cutting his work short, he would not have had this opportunity to achieve success, grow in confidence, and develop meaning for anti-differentiation. Time is often viewed as the rarest commodity in classrooms. However, if educators can learn to focus on students' needs, rather than time shortages, they might be better equipped to provide their students with opportunities for risk taking and to help students develop rich, conceptual understandings of significant mathematics. Such endeavors could be time well-spent, potentially saving time in the future as students might not require later remediation and review on mathematics they are expected to have learned already.

8.4 Teachers Willing to Risk

Lastly, and perhaps most importantly, teachers need to be more willing to take intellectual risks themselves (NCTM, 2007). Becoming a student-centered teacher places great demands on a teacher's skill, knowledge, and confidence. Teachers must know mathematics to such an extent that they can create, adapt, and implement appropriate tasks for their students. This requires knowledge not only of the structure and function of mathematics, but application strategies and ways to help students become problem-posers and consensus builders in making sense of new mathematics. For those teachers who are accustomed to simply telling their students what information to know and expect their students to take notes, repeat the procedure several times in homework, and then to grasp the concept, taking a risk in going outside their comfort zone and altering their teaching style might lead to increased student learning and efficacy. As teachers become familiar with types of questions that help elicit student responses indicating understanding and use such tactics on a daily basis, they will provide their students with additional opportunities to exercise agency in developing understanding of mathematics. As teachers themselves become problem-posers, problem-solvers, abstract thinkers, and then use precious time in behalf of their students, they can help their students delve deeply into mathematics and in turn become problem-posers, problem-solvers, and mathematical thinkers.

CRT responds to the research question of what mathematical risks students take by outlining some risks students take in learning mathematics. CRT claims that students are capable of dealing with mathematical uncertainties and find enjoyment in working through the uncertainties accompanying difficult mathematics. In grappling with uncertainty, students find opportunities to become problem-posers and develop personal understanding of the purposes and meanings of significant mathematics. CRT offers suggestions for mathematics teachers to improve learning and teaching in the mathematics classroom by detailing specific conditions that contributed to a risk-friendly classroom environment encouraging of student risk-taking in this study. CRT responds to the research question by illustrating how taking risks influences student creation of mathematical meaning.

Appendix A: Student Perspectives Survey

Note: Two questions, numbers 13 and 14, dealt with technology usage in the classroom and were not relevant to this study on student risk. Therefore, student responses to questions number 13 and 14 are excluded from the survey.

	Background Information, Standing, Major		**Question 1: List three qualities of an excellent mathematics learner**	
Student	**Pre-course**	**Post-course**	**Question 1 Pre-course**	**Question 1 Post-course**
Andrew	Junior Bioinformatics		Persistent, Open Mind, Does Many Practice Problems	
Carina	Sophomore Mathematics Education	Sophomore Mathematics Education	Open minded and abstracts thoughts, Patient and flexible, Consistent when solving exercises	Abstract minded Systematic Flexible with solving problems
Kameron (Kam)	Junior, Mechanical Engineering, ACT: 25 Math 112H Pretest: 90 Algebra 1: A Algebra 2: B Trig: C Geometry: B	Junior, Mechanical Engineering	1. Enjoys Mathematics. 2. Diligence, persistence. 3. An inquisitive mind	Persistence, curiosity, desire to learn, open mind
Mark		Sophomore, Bio-informatics/ computer science		Ability, inquiry, dedication
	Question 2: Which of the qualities you listed above, do you feel is your strongest? Please Explain.		**Question 3: Which of the qualities you listed above, do you feel is your weakest? Please explain.**	
Student	**Question 2 Pre**	**Question 2 Post**	**Question 3 Pre**	**Question 3 Post**

Andrew	Persistence – I try until I fully understand.		Open Mind – it's easy to get stuck with false beliefs about math.	
Carina	Patient and flexible because I never get frustrated with a problem and if it doesn't work, I try different manners to solve it.	Flexible: I always find different ways of solving a problem, even though a pattern is given.	Consistent when solving exercises because I don't have a daily routine for solving exercises.	Systematic: I lack when keeping track of homework and study for test, because I think "I can do it." I know that if I'd been more consistent and systematic I could do better in math.
Kameron (Kam)	"Enjoys mathematics"- I enjoy mathematics more than most of my peers. This increases my motivation and changes math from work to play.	Curiosity, learning new things is very exciting to me. I enjoy learning.	"Diligence, persistence." I tend to be lazy and give up when I fail at something.	Persistence, I can be lazy sometimes and give up occasionally.
Mark		Dedication. I usually keep at a task until I finish it.		I don't think I have a natural gift for mathematics.
	Question 4: What does it mean to be a successful mathematics learner?		**Question 5: Describe an optimum classroom environment for learning mathematics. Why are these conditions optimum? What would be the practices within this environment?**	
Student	**Question 4 Pre**	**Question 4 Post**	**Question 5 Pre**	**Question 5 Post**
Andrew	To have the		1. Introduction	

	desire to learn and the work ethic to back it up.		to a subject from the basics first, 2. A real world explanation followed by equation explanation, 3. Student practice with homework and quizzes, 4. Problems that challenge but don't baffle students. Often teaches will give assignments without giving students proper preparation in an attempt to "challenge" their minds when it really only waste learning time. Just give the information the students need and challenge them with a faster pace rather than held back information.	
Carina	That once you get the concepts and general patterns, you can apply them in the different situation a problem can be	To get understanding about math topics and be able to explain them to others.	In silent and a neutral temperature, so you don't get cold or too worm to lose focus. With people willing	A lot so you don't get tired when reading and solving problems. Group work because 2 or more heads think better and

	planted.		to learn and share ideas on how to solve the exercises so we can all feel less frustrated in the process.	help you to get different manner of solving problems. Share T.A.s and or professors willing to help you if you have questions.
Kameron (Kam)	It means to understand mathematics principles and their applications in a memorable and usable way.	To explore and try to come up with ideas and solutions. To think creatively.	The teachers and students have respect for one another and feel comfortable asking questions and participating. Multiple approaches are discussed so that each student can decide what works best for them. There would be a lot of participation. Students would have opportunities to teach so that they can remember and understand the principles more fully.	This classroom has been optimum for me, because I can discover math myself and I feel a sense of accomplishment and am able to remember the techniques better. We are able to work in groups and teach each other.
Mark		To have a mastery of the concepts and understand the procedures. Why we do what we do to get an answer.		Small class size. Relatively smaller room. Lots of whiteboards. Assignment is given prior to class everyone

				goes up to a white board. Problems are worked out. Students help one another. Teacher gives explanation. Lecture is given on homework. Students go to board to work on problems.
	Question 6: What is mathematics?		**Question 7: What are the purposes of mathematics?**	
Student	**Question 6 Pre**	**Question 6 Post**	**Question 7 Pre**	**Question 7 Post**
Andrew	Putting real world quantities and models into a form that can be manipulated.		To solve real world problems in a quantitative manner.	
Carina	The science that studies the manners on how we can measure the world and the way we can solve problems.	It's the science that studies the behavior of the world in terms of numbers and analytical thinking.	Help people to develop a critical mind and an abstract mind to understand concepts and situations.	Help you to develop on abstract and creative way of see the world. It helps you to apply different solutions to different situation problems. If you know math, it'll be easier to learn any other subject.
Kameron (Kam)	Mathematics is the way to describe and understand processes numerically.	Numerical description of the natural world. And also an abstract exploration.	The purpose of mathematics is to allow greater and more precise understanding of our natural world. Another purpose is to	To describe the natural world, and expand the mind and creativity.

			expand and develop the mind.	
Mark		The study of math. Math is the study of number theories.		To expand the minds and find practical solutions to the world's problems.
	Question 8: What do you like most about mathematics? Please explain.		**Question 9: What mathematics have you most enjoyed learning? Please be specific and explain why you find these particular topics engaging.**	
Student	**Question 8 Pre**	**Question 8 Post**	**Question 9 Pre**	**Question 9 Post**
Andrew	There are many problems that are nearly impossible to solve without mathematics.		Calculus – Even though I took calculus and only got a C+ I actually enjoyed calculus the most because of its challenges and possibilities. Other math subjects had little real world application other than simple problem solving. After taking calculus I really look forward to math learning.	
Carina	I like that I feel I can read better, explain myself better, and see the world from a	The fact that I can make sense of numbers. I also like math because by doing so, you	Algebra, because I can use a lot of logic applicable to a problem.	Calculus and algebra because with both you can solve many of the physics

	simplify point of view when I study math.	exercise your brain and help you to be a reasonable person.		problems such as the behaviors of a car etc. I love working with numbers more than with figures and angles such as geometry.
Kameron (Kam)	I like the fact that there are usually definite answers and that it has so many applications in the physical world.	Its application to the physical world, because I enjoy physics.	I enjoyed a great deal of trigonometry. Specifically: complex numbers, identities, and vectors. The reason I enjoyed these is because they were sometimes challenging to understand. I also enjoyed factoring in algebra.	Trig and Calculus. Because they are so useful in application to the physical world. I.e., velocity, acceleration, work, energy.
Mark		A challenge. Solving a puzzle, a riddle.		Interest problems. Because they have real life application for me.
	Question 10: What do you find least appealing about mathematics? Why?		**Question 11: List three necessary qualities of an excellent mathematics teacher.**	
Student	**Question 10 Pre**	**Question 10 Post**	**Question 11 Pre**	**Question 11 Post**
Andrew	The only thing about calculus I did not like was the teacher I had for calculus (here at BYU).		A good teacher teaches students in plain English first, and equations	

	He was the worst math teacher I have ever had. He seem more intent on showing you how smart he is (after 40 years of teaching calculus), than teaching students. I would have enjoyed it better if not for him. I spent more energy trying to figure out his puzzling way of communication than on calculus itself. As for the text book it seemed like that teacher but in written form. I would have been better off buying a better textbook and teaching myself. I would be willing to bet that I would have done better on the final it that were the case.		second. Equations will make little sense until the idea is understood. The teacher will focus mostly on solving problems in class after the main ideas have been expressed. The teacher will show multiple ways of looking at and solving the problems. The problems would be better if there are many more slightly challenging problems and fewer very challenging ones. The problems would focus on variety rather than difficulty. Often a teacher would define a challenging problem by how much the student needs to figure out on their own, but that approach only challenges intelligence and	

			not math competency. Many, many problems should be provided that build on each other. No problem is challenging if all the building block ideas are fully understood and that is the point isn't it? The teacher will have all homework planned ahead of time and on a syllabus before the first day of class. They will not say something stupid like, "we'll be flexible on the homework schedule". Any college math teacher should be competent and prepared enough to know how long subjects take to learn and have a plan from the beginning so students know what to expect. Also, five classes a week is far too much.	

			The reality is that students learn mostly on their own, and class time is only to clarify and solidify the text. Five days a week only make things tedious. The primary reason I took this section was that it was only MWF.	
Carina	[left blank]	That sometimes it's hard to find a connection with the real world. You have in most of the cases an answer for a problem, but that is not always the car in real life.	Charismatic, Funny, Integral	[Allowment?] Willing to give different pattern of solving a problem Flexible with your mistakes
Kameron (Kam)	I am sure there is something, but I simply can't think of it right now.	Definition of a limit. I don't find it useful personally.	1. An ability to clearly explain mathematical principles in multiple ways. 2. Knowledge of the material. 3. An understanding of teaching and learning techniques	Knowledge of math, communication skills, asks right questions
Mark		Long hours of homework. Obvious reasons.		Love/caring Ability to come down to the learner's level. Patience.
	Question 12: Please describe the teaching style of your best		**Question 15: What do you feel are the responsibilities**	

	mathematics teacher.		of a student in this course?	
Student	**Question 12 Pre**	**Question 12 Post**	**Question 15 Pre**	**Question 15 Post**
Andrew	Like that described above.		Do the homework, read the text, and work until I understand what is presented.	
Carina	With humor and creativity my professor exposed most of his classes, trying to show that math is not a hard killer topic. He also helped us to develop the critical thinking when solving a problem.	Funny, showing math like a fun and exciting topic Task given so you can think and analyze a problem in different ways Creative: every time it was presented in a problem in a different way	Listen carefully to the instructions and be willing to help others and listen to the other members of the group.	Keep track of everything and do the homework. Ask questions. Participate within your groups by pointing your ideas.
Kameron (Kam)	She explained principles very clearly and in multiple ways. She used humor in teaching. She had an apparent passion for the subject.	Asked the perfect questions to help us answer our own questions and think in ways that would help us figure problems out.	The student's responsibilities are to be open-minded, attentive, and hard-working. I also consider it a responsibility to be involved.	To have an open mind, to be willing to teach and learn from everyone and contribute to the class and group.
Mark		Lots of interaction with the students, willing to work with the students' strengths and weaknesses.		To do their best
	Question 16: What do you feel are the responsibilities of a teacher in this course?			

Student	**Question 16 Pre**	**Question 16 Post**		
Andrew	Like that explained in question 11.			
Carina	Show us the different manners on how we solve a problem using calculus.	Answer the question in different ways. Give the patterns and [reasons?] that will help you to solve problems. Provide you with useful tools for solving problems. Prepare a good final.		
Kameron (Kam)	The teacher's responsibilities include allowing for 1-on-1 time if possible, providing classroom structure, allowing interaction and participation, and explaining principles clearly and in multiple ways.	To be patient as students figure things out. To ask good questions.		
Mark		To work with the students. Even the [weaks?]		

Appendix B: Code Definitions and Categories

Code Definitions

Codes are grouped alphabetically first by categories, then within categories.

Acknowledge peer	Speaker recognizes that something a peer has said is ingenious, creative, and/or correct	C – acknowledge peer
Appeased	Student who once was insistent about a particular thing is no longer adamant	C - appeased
Comparing with peers' progress	Asking a peer to compare progress with the speaker's progress (compare progress)	C - compare
Confirmation	Speaker is confirming something to peer	C - confirm
Confusion	A student demonstrates characteristics of being confused by mathematics or peer	C - confusion
Contribution	Speaker is making an effort to contribute to the group's progress	C - contribute
Discrepancy	Student has encountered a discrepancy between their intuition or calculated work and that of something else (another person, calculator, etc…)	C - discrepancy
Hesitation	Speaker demonstrates a desire to pause before pursuing an idea or actively continuing in mathematical work	C - hesitation
Lack of knowledge	Student admits that he/she does not know something	C – lack knowledge
Processing thoughts	Speaker seems to be more focused on processing information than articulating anything	C - processing
Peers' progress	Interested in a peers' progress	C – progress
Recognition	Something in the speaker's tone/word usage implies recognition of some mathematical principle or idea	C - recognition
Resignation	Speaker has given up on something and resigned him/herself to no longer worry about it.	C - resignation
Self-answered	Some question or concern raised by the speaker	C – self-answered

	has been answered/address by the speaker	
Self-correction	Speaker corrects himself while speaking aloud	C – self-correct
Seek approval	Speaker is looking to peers to affirm or validate what they've said	C – seek approval
Seeking comprehension	Speaker wants to be sure that he/she is being listened to and understood: "you know what I'm saying"	C – seek comprehension
Peers' solution	Interested in a peers' solution	C - solution
Understate	Speaker is understating the significance, importance, or level of good work performed; like unto modesty	C - understate
Adventure	Speaker views self as engaged in some mathematical adventure	E - adventure
Agree/agreement	Speaker is agreeing with a peer about something	E - agree
Belief	Speaker indicates acceptance of an idea presented by peer	E - belief
Believing game	Speaker indicates for peer to continue with hypothesis/explanation even though it is evident they do not completely agree or are not confidant or comfortable with peer's statement	E – believing game
Confidence	Speaker is demonstrating strength and confidence in topic at hand (whether proposed by self or peer)	E – confidence (self), (peer)
Defeated	Speaker's actions or tone indicate a pessimistic, sad attitude of someone who feels like they cannot achieve what he/she set out to do	E - defeat
Dismiss	Student decides to disregard something in favor of another way of thinking or approaching a problem	E - dismiss
Doubt	Speaker implies disbelief in statement or claim made by a peer	E - doubt
Enjoyment	Speaker seems/appears to be enjoying his/herself and/or the mathematics he/she is doing	E - enjoyment

Excitement	State of being eager and anticipatory of mathematical activity; moves into action (?); speaker demonstrates keen interest in work or task	E - excitement
Intuition	Student refers to a gut feeling about something being wrong or right	E - intuition
Satisfaction	Tone or words provide evidence that student is pleased with self or other and mathematical progress.	E - satisfaction
Self-doubt	Speaker indicates lack of confidence in verbally shared mathematical work (mathematical self-doubt)	E – self-doubt
Support/encouragement	Speaker is showing some sort of sympathy or encouragement for a peer to continue with their idea or explanation	E - support
Unappeased	Speaker has some unresolved issue; may persist in searching for a particular piece of information or explanation because one previously provided was not sufficient enough to cause speaker to develop meaning/understanding/confidence, or may ignore and move on.	E - unappeased
Addition	Combining to equations through addition	MC - addition
Algebra	Speaker is referring to or performing algebra	MC - algebra
Algorithm	Discussing the algorithm to some mathematical idea -integrating	MC - algorithm
Bound	Speaker is referring to some boundary laid on a mathematical entity (a bounded function, graph, etc…)	MC - bound
Constant	Reference to some constant term in a mathematical equation -of integration	MC - constant
XY-Coordinates	Reference (direct or indirect) to points in x, y coordinate form (x, y)	MC - coordinates

Definition	Speaker is referring to the essence of what a particular term is or means	MC - definition
Geometry	Speaker is referring to or using geometric principles	MC - geometry
Height	Referring to some mathematical quantity that represents height of something	MC - height
Intersections/ intersection points	Student work is referring to (indirect or direct) the intersection(s) of two data sets, typically linear equations/lines	MC - intersection
Labels	Speaker is referencing an organizational method of labeling functions, graphs, etc.	MC - label
Midpoint	Speaker referring to the middle value s.t. abs(a-m)=abs(m-b)	MC - midpoint
Negative	Some notable value is negative (i.e. less than zero) -people	MC - negative
Parabola/parabolic	Referring to a parabola or a graph whose curve is parabolic	MC - parabola
Parallelogram	Referral to a parallelogram (closed quad where both pairs of opposite sides are parallel)	MC - parallelogram
Problem-solving process	Speaker is referring to the process of coming to a solution (i.e. talking about needing to explain how they came up with their solution	MC – problem solving
Rate	Referring to rate of change of some quantity in the TLT	MC - rate
Reality	Reference to some real world property that affects the way students are approaching/working through the problem	MC - reality
Rise	Referring to the vertical (unit) increase of a given linear function	MC - rise
Run	Referring to the horizontal (unit) increase of a given linear function	MC - run

Slope	Referring to the rate of change of a linear function.	MC - slope
Slope-intercept form	Referring to writing a linear function as $y=mx+b$	MC – slope-intercept form
Substitution	Speaker is referring to or utilizing substitution	MC - substitution
Subtraction	Combining equations through subtraction/taking their difference	MC - subtraction
Translation	The shift or slide of the graph of a function in one direction -vertical -horizontal	MC – translate (v), (h)
Trapezoid	Referral to a trapezoid (closed quad where at least one pair of opposite sides are parallel)	MC - trapezoid
Triangle	Speaker is referring to or using triangles	MC - triangle
Unknown	Speaker referring to some variable in an equation whose value is not known	MC - unknown
x-value	Student is concerned with finding the x-value of some point in space	MC – x-value
y-axis	Referring to the y axis; i.e. the line $x=0$	MC – y-axis
y-intercept	The y-value where a function crosses the y-axis, or the y-coordinate when $x=0$	MC – y-intercept
Assert	Speaker is making a direct claim with *great* confidence and persistence/insistence	MR - assert
Beginning	The start of some big idea that will direct future work	MR - beginning
Benefit	Student articulates benefit of pursuing some particular path	MR - benefit
Challenge another	Speaker is disagreeing with peer; sometimes implying that he/she expects peer to provide reasoning and justification for the claims previously made	MR - challenge

Conclusion	Speaker reaches some climactic point that was the focus/apex of the preceding dialogue	MR - conclusion
Counter-hypothesis	Speaker contradicts a peer b/c of belief peer is incorrect; in opposition of one student's hypothesis, a different hypothesis is proposed	MR – counter-hypothesis
Difficulty	Student recognizes the level of difficulty of a given task/work -easy -difficult -acknowledge	MR – difficulty (acknowledging, easy, hard)
Extension	Speaker is taking an idea immediately previously discussed and adding/building on it	MR - extension
Hypothesis	Speaker is articulating brainstorming thoughts about various mathematical routes that could potentially progress towards a solution	MR - hypothesis
Increased understanding	Indication that the speaker has come to an understanding of something previously misunderstood	MR – increased understanding
Justification	Reasoning provided that demonstrates the reasons behind making a claim; strong evidence -empty (provides no real reasoning)	MR - justification
Mathematical meaning	Speaker demonstrates a desire or competency in developing meaning for the mathematical work (i.e. symbols, equations, numbers, etc.) in which they are engaged -contextual -numerical	MR - meaning
Mixed understanding	Speaker indicates that he/she has preliminary comprehension but not a sound understanding	MR – mixed understanding
Need	Speaker recognizes need for some thing -mathematical meaning -background understanding -numbers	MR – need (meaning, background, numbers, equations, algorithm, formula)

	-equations -algorithm -formula	
Different perspective	Speaker(s) recognize that they are approaching the problem differently than their peers	MR - perspective
Plan	Speaker stating intent travel a particular mathematical path	MR - plan
Proposition/propose new idea	Introduction of a big mathematical idea that has not previously been discussed; implementation of such an idea would alter the mathematical course the students are on	MR - propose
Seek meaning	Speaker is actively looking to find the meaning of mathematical work -contextual	MR – seek meaning
Seek understanding	Speaker is trying to ensure that participant in dialogue (whether self or peer) is conceptually familiar with the topic of conversation	MR – seek understanding
Uncertainty	Referral to the fact that speaker does not know the outcome of a mathematical avenue – either they admit they don't know what they are doing is right or where it will lead them, or they imply lack of confidence in their work -method -truth of statement	MR - uncertainty
A(p)	$A_p = bh$	MT – A(p)
A(t)	$A_t = \frac{1}{2}bh$	MT – A(t)
Anti-derivative	Speaker is referring to the anti-derivative of some function	MT – anti-derivative
Application	Referring to the application of some mathematical idea/principle	MT - application
Area	Speaker is referring to the idea of, or finding/using the area of some region.	MT - area

Assessment	Speaker is engaging in some form of assessment of a peer's mathematical work -situational… - specific to the situation (I think this would be the only one)	MT - assess
Body use	Speaker is using hand gestures, etc to demonstrate something	MT - body
Calculate	Student is doing mathematical calculations	MT – calc
Calculator	Student use of or referral to a calculator	MT - calculator
Chain rule	Speaker is referring to or using the chain rule of anti-differentiation (i.e. the "anti-chain rule")	MT – chain rule
Characteristic(s)	Speaker recognition or use of defining elements of mathematical objects or terms	MT - characteristic
Checking work	Student is re-checking somebody's work	MT - check
Combining	Combining two equations/pieces of information to make one -equations -information	MT - combine
Comparing functions	Speaker is comparing two different functions to one another in some way	MT – compare functions
Conditional	Student declares or recognizes some constraint in something (whether mathematical work or emotional well-being…)	MT - conditional
Elaborating	Speaker is elaborating on his/her work, explaining in-between steps, etc…	MT - elaborating
Equations	Explicit referral to the presence of or need for equations (in the general sense)	MT - equations
Equivalence	Speaker referring to or recognizing that a number of pieces are equal in size, length, area, volume, etc.	MT - equivalence
Exact	Speaker is indicating precision and exactness of something	MT - exact
Example	Student is using or referring to an example to	MT - example

	discuss some current principle or thought	
Formula	Speaker is using some plug-and-chug formula in their work	MT - formula
Graph	Speaker is referring (directly or indirectly) to some graph -shape -characteristics	MT – graph (s, ch)
Increase	Some quantitative value is getting larger -line length	MT – increase (or see "line length")
Work-in-progress	Student acknowledges that his/her or peer's work is not a polished, complete version but still something in-the-works	MT – in-progress
Integral	Referring to the resulting equation of indefinite integration of a specific equation/function -definite integral	MT - integral
Integration	Speaker is referring to or using integration: the act of finding an integral of a specified function	MT - integration
Line length	Speaker is making a statement about the length of the line at a particular time -0 -200 -increase -decrease	MT – length (_)
Net flow	Reference to the overall effect on the line length (i.e. 20 people going in, 4 going out, net flow is 16 in)	MT - net flow
Numbers	Referral to the presence or need of numerical information (in the general sense)	MT - numbers
Pieces	Speaker is breaking down a larger region into smaller pieces	MT - pieces
Prior experiences	Speaker is claiming a connection to some previous experience	MT – prior (TLT, QR)

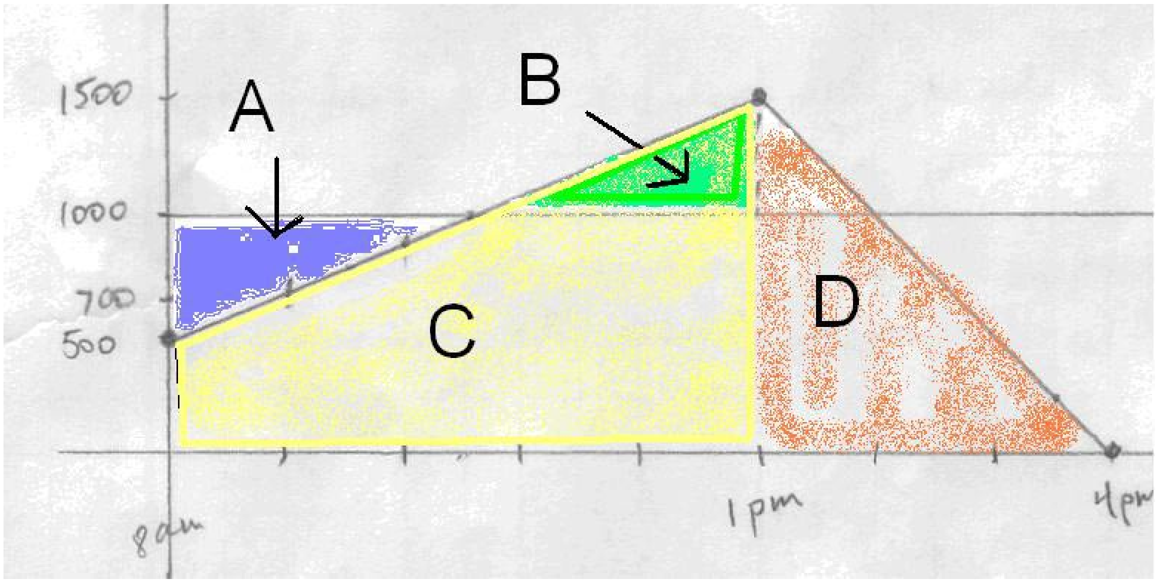
Steady decrease	Referring to fact that some quantity is decreasing at a constant rate	MT – rate (steady decrease)
Steady increase	Referring to fact that some quantity is increasing at a constant rate	MT – rate (steady increase)
Rate arriving	Referring to the rate of change of people arriving at the ticket office; incoming	MT – rate arriving
Rate leaving	Referring to the rate of change of people being served at the ticket office; outgoing	MT – rate leaving
Comparing rates	Speaker is making some comparison between the rates of two or more linear functions	MT – rates (compare)
Recitation of fact/formula	Students spits out a memorized fact (like area of a triangle)	MT - recitation
Region	A two-dimensional closed space; speaker typically refers to finding the area of a given region	MT - region
Region A	See pictures	MT – region A
Region B	See pictures	MT – region B
Region C	See pictures	MT – region C
Region D	See pictures	MT – region D
Region F	See pictures	MT – region F
Region G	See pictures	MT – region G
Region H	See pictures	MT – region H
Rephrasing	Speaker is restating something that has previously been discussed at the table, typically immediately previously, using slightly different language/approach	MT - rephrase
S(t)	Andrew's graph of y=100x^2 – 500x + 600	MT – S(t)
Seeking clarification	Speaker is making a statement or comment towards clarifying what another has said; typically in the hopes of increasing understanding or comprehension	MT – seek clarification

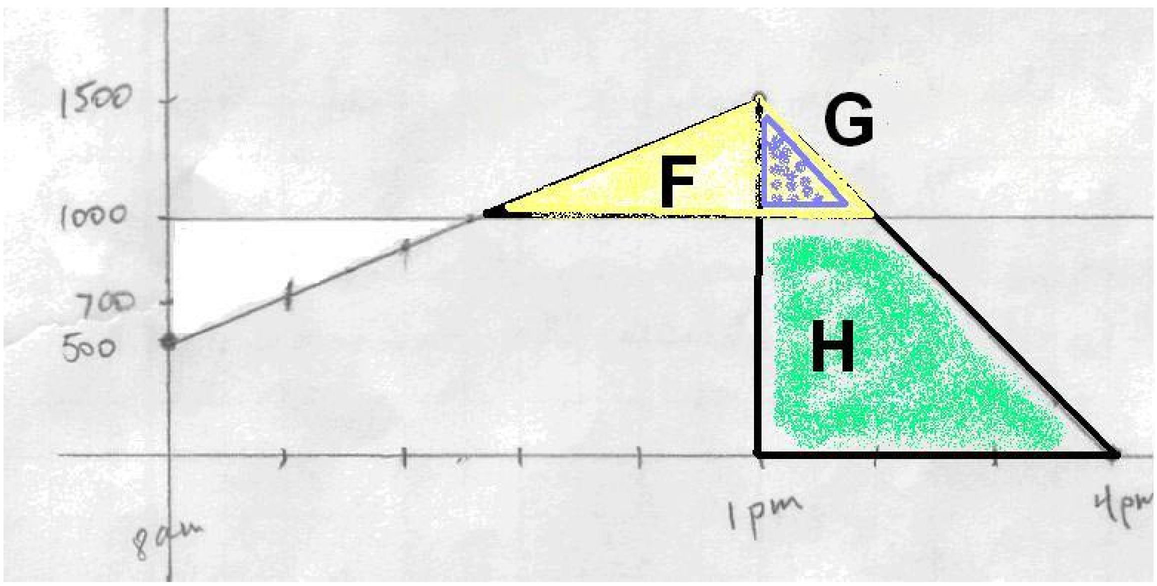
Seek elaboration (explanation)	Speaker is seeking an elaborated explanation of a peers' process or method	MT – seek elaboration
Seek method	Speaker is looking for a process to do some specific mathematical activity and thereby achieve some mathematical goal	MT – seek method
Solve	Student is solving for some unknown value	MT - solve
Trace (function)	Speaker is referring to or using the trace feature on a graphing calculator	MT - trace
Trial and error	Student is employing or recounting use of trial and error in coming to some part of their mathematical work/solution	MT – trial and error
Un-equivalent	Speaker compares two elements and claims that they are not equivalent	MT – un-equivalent
Working aloud	Student verbalizing their work either as they are thinking or brainstorming it, do it, or when reproducing it	MT – work aloud
Student work	Reference (direct or indirect) is made to a student's written or verbal mathematical work with numbers/operations -calculations -graph	MT – work(calc, graph)
Y1	$y_1 = 200x + 500$	MT – y_1
Y1a	Y= (200x^2)/2 + 500x + b	MT – y_{1a}
Y2	$y_2 = -500x + 4000$	MT – y_2
Y3	$y_3 = 1000$	MT – y_3
Y4	$y_4 = 0$	MT – y_4
Y5	$y_5 = -1000$	MT – y_5
Y6	$y_6 = 100x^2 + 500x + 600$	MT – y_6
Y7	$y_7 = 100x^2 - 500x + 600$	MT – y_7

Y8	$y_8 = 1000x$	MT – y_8
Y9	$y_9 = -1000x$	MT – y_9
-500	When 500 people are entering and 1000 are leaving, the net flow is –500	T – (-500)
10:30 am		T – 10:30am
1000		T - 1000
10 am		T – 10am
1500		T - 1500
1 pm		T – 1pm
200		T - 200
4 pm		T – 4pm
500		T - 500
600		T – 600
625		T - 625
8 am		T – 8am
9 am		T – 9am
Given Information	Information that was provided in the text of the TLT	T - given
Initial conditions (starting value)	Referring explicitly to the fact that the task states that at 8 am there are 600 people in line, i.e. (0, 600) is a data point of interest	T – initial condition
Interval A	8-10:30am	T – interval (A)
Interval A+	10-10:30	T – interval (A+)
Interval B	10:30am-1pm	T – interval (B)
Interval C	8am-1pm	T – interval (C)
Interval D	1pm-4pm	T – interval (D)

Part A	Sketch a graph of the rate at which students arrive at the ticket office as a function of time	T – part A
Part B	At first, because students are served at a rate greater than the rate at which they arrive, the line decreases in length. Does it disappear before students begin arriving at a rate greater than the rate at which students are being served? About when does the line again form or begin to lengthen?	T – part B
Part C	About when is the line the longest? About how many students are in line then? About how long would the last student in line at that instant have to wait to be served?	T – part C
Part D	About when does the line finally disappear?	T – part D
Part E	About how many students are served by the ticket office that day?	T – part E
Part F	Sketch a graph of the length of the line as a function of time.	T – part F
Task referral – direct	Direct referral to some element of the Ticket Line Task	T – referral (direct)
Task referral – indirect	Indirect referral to some element of the TLT	T – referral (indirect)
Truncated A	Region A, but cut off at 10 instead of 10:30	T – region (A-)

Region Definitions





Appendix C: Sample Episode Report

Sample episode report for the last 18 minutes of annotated, transcribed video:

Category	**Code**	**Frequency**
Collaboration	acknowledge peer	2
Collaboration	compare	15
Collaboration	confirm	21
Collaboration	discrepancy	4
Collaboration	hesitancy	6
Collaboration	lack knowledge	4
Collaboration	processing	2
Collaboration	resignation	1
Collaboration	self-answered	2
Collaboration	self-correct	1
Collaboration	seek approval	13
Collaboration	seek comprehension	6
Emotion	agree	5
Emotion	belief	14
Emotion	believing Game	15
Emotion	confidence	16
Emotion	dismiss	2
Emotion	doubt	2
Emotion	enjoyment	2
Emotion	excitement	13
Emotion	intuition	4
Emotion	satisfaction	3
Emotion	self-doubt	10
Emotion	support	46
Mathematical Concepts	addition	3
Mathematical Concepts	algebra	2
Mathematical Concepts	algorithm	2
Mathematical Concepts	bound	1
Mathematical Concepts	constant	6
Mathematical Concepts	intersection	1
Mathematical Concepts	negative	9
Mathematical Concepts	parabola	1
Mathematical Concepts	reality	3
Mathematical Concepts	substitution	3
Mathematical Concepts	subtraction	3

Mathematical Concepts	translate	2
Mathematical Concepts	y-intercept	8
Mathematical Risks	beginning	5
Mathematical Risks	benefit	3
Mathematical Risks	challenge	4
Mathematical Risks	conclusion	12
Mathematical Risks	counter-hypothesis	3
Mathematical Risks	difficulty	2
Mathematical Risks	extension	5
Mathematical Risks	hypothesis	27
Mathematical Risks	increased understanding	9
Mathematical Risks	justification	14
Mathematical Risks	meaning	40
Mathematical Risks	mixed understanding	1
Mathematical Risks	need	3
Mathematical Risks	perspective	7
Mathematical Risks	plan	11
Mathematical Risks	propose	13
Mathematical Risks	seek understanding	4
Mathematical Risks	uncertainty	29
Mathematics from Task	anti-derivative	18
Mathematics from Task	application	1
Mathematics from Task	area	11
Mathematics from Task	check	15
Mathematics from Task	combine	12
Mathematics from Task	compare functions	1
Mathematics from Task	elaborating	38
Mathematics from Task	equations	5
Mathematics from Task	equivalence	5
Mathematics from Task	example	2
Mathematics from Task	formula	1
Mathematics from Task	in-progress	2
Mathematics from Task	integral	9
Mathematics from Task	integration	1
Mathematics from Task	length	13
Mathematics from Task	length- decreasing	1
Mathematics from Task	length- increasing	2
Mathematics from Task	length- number	20
Mathematics from Task	net flow	8

Mathematics from Task	pieces	13
Mathematics from Task	prior- TLT	3
Mathematics from Task	prior- other	7
Mathematics from Task	rate arriving	9
Mathematics from Task	rate leaving	13
Mathematics from Task	rate- compare	1
Mathematics from Task	region A	4
Mathematics from Task	region B	6
Mathematics from Task	region F	1
Mathematics from Task	region G	1
Mathematics from Task	rephrase	27
Mathematics from Task	seek clarification	21
Mathematics from Task	seek elaboration	2
Mathematics from Task	seek method	12
Mathematics from Task	solve	1
Mathematics from Task	trace	5
Mathematics from Task	work	2
Mathematics from Task	work aloud	26
Mathematics from Task	world- calculate	20
Mathematics from Task	work- calculator	16
Mathematics from Task	work- graph	17
Mathematics from Task	y1	7
Mathematics from Task	y1a	3
Mathematics from Task	y2	1
Mathematics from Task	y3	6
Mathematics from Task	y6	7
Mathematics from Task	y7	17
Mathematics from Task	y8	6
Mathematics from Task	y9	4
Speaker	Andrew	157
Speaker	Carina	56
Speaker	Kam	63
Speaker	Mark	12
Speaker	Rebecca	4
Task	10:20 am	8
Task	10:00 am	4
Task	1500	1
Task	1:00 pm	4
Task	250	1

Task	600	2
Task	625	4
Task	8:00 am	1
Task	9:00 am	1
Task	given	1
Task	initial condition	9
Task	interval A	2
Task	interval A+	6
Task	interval A-	3
Task	interval B	2
Task	interval C	1
Task	referral- direct	1
Task	referral- indirect	2

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